Stochastic Dynamics for Accelerators

A. Pauluhn
Deutsches Elektronen-Synchrotron DESY, Notkestr. 85
2000 Hamburg 52, Germany

Abstract

In this report we introduce several approaches to stochastic problems in accelerator physics. The first part of the present work treats the concepts of stochastic differential equations (SDEs) and of Fokker-Planck equations (FPEs), whereas in the second part we concentrate on discrete models and investigate a method of calculating density functions via stochastic mappings.

I. INTRODUCTION

The motion of particles in an accelerator is strongly influenced by various stochastic effects such as ground motion, power supply ripples, noise caused by the quantum emission of synchrotron radiation and explicit noise in the rf system. A good description of external noise, i.e. of the influence of a great number of nearly uncorrelated, rapidly fluctuating random effects on a system, is the Gaussian white noise process (GWN) $\xi(t)$. This process has the properties $<\xi(t)>=0$ and $<\xi(t)\xi(t')>=\delta(t-t')$. Another advantage of using the white noise concept is the fact, that for such noise "inputs" the resulting process is a Markovian process. Markovian processes are those, whose future depends only on the present and not on the past ("without memory"). We present some methods of treating differential equations which include stochastic quantities and calculate moments and density functions. In the case of nonlinear, time dependent coefficients in the describing equations very few results exist, and good numerical tools become necessary. We investigate integration schemes for SDEs and apply them to several examples like a simple model of the beam-beam interaction and the synchrotron motion. For the study of the synchrotron motion we also chose a kind of complementary access to stochastic systems as opposed to stochastic differential equations, the Fokker-Planck equation.

The second part of this study is concerned with a discrete approach to a nonlinear damped stochastically excited system like particle motion in an e$^-$-storage ring. We present an algorithm for computing the density function and follow its evolution in time. By tracking particles in a discretized phase space we compute a stochastic matrix as a time propagator for the density function, again making use of the fact that we describe a Markov process. We study simple models of the beam-beam effect and compare our results with results obtained via usual tracking techniques.

II. MODELLING STOCHASTIC EFFECTS WITH SDES

Including stochastic effects in the equations of motion of a dynamical system, the system variables become stochastic processes, i.e. time dependent random variables. In our investigations we start from equations of the form

$$\dot{\vec{x}} = \vec{f}(\vec{x}) + \underline{g}(\vec{x})\vec{\xi}(t) \quad ,$$

where $\vec{\xi}(t)$ describes the noise process and has the properties of the GWN. The first term of the right hand side gives the deterministic part of the equation, and the second introduces a diffusion component.

A. *Numerical Solution*

For a numerical approach to SDEs, one performs the following steps:
i) Taylor expansion of the approximate solution in the stepwidth h
ii) model the noise process and, if necessary, functionals of it
iii) simulate lots of realizations for averaging.

Difficulties in the simulation procedures are for example contained in the second point of the listing: how can such an irregular process as WN, or higher order functionals of it be modelled? In simulations these expressions have to be substituted by simple functions of random vectors, so that they yield the same moments up to a given order, which is often quite CPU-time consuming. Therefore one is limited to algorithms of low order [1],[2].

B. *Examples:*

a. *Beam-Beam Model:*

As an example for a time-dependent potential we made calculations for a one-dimensional SDE with beam-beam kick, damping and noise for a simple beam-beam interaction model (round beams, weak-strong approximation).

The equation reads:

$$\ddot{x} + \alpha\dot{x} + \omega^2 x + f(x,s) = \sqrt{d}\cdot\xi(s),$$

with

$$f(x,s) = -8\pi\xi_{bb}\left(\frac{1-e^{-\frac{1}{2}x^2}}{x}\right)\delta_p(s),$$

where $\delta_p(s) = \sum_{n=-\infty}^{\infty}\delta(s-nL)$ and $\sqrt{d} = \sigma\omega\sqrt{2\alpha}$. We used a ring of length $23.25m$ with a Q-value of 3.7 and a strong beam-beam parameter of $\xi_{bb} = 0.07$. The damping time was taken to be 1000 turns. We chose the weight of

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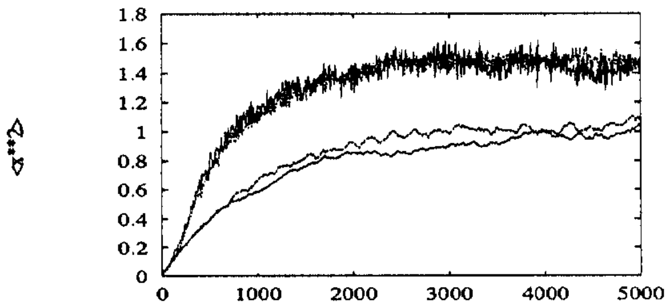

Figure 1: $< x^2 >$ vs. no of turns, linear case (lower curves) and with beam-beam kick force (upper curves).

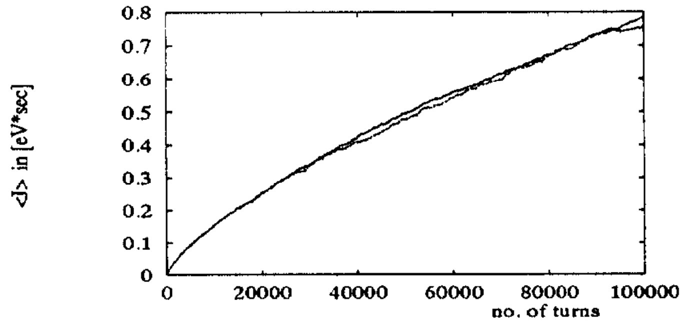

Figure 2: $< J >= \pi < \epsilon >, \phi_0 = 0.1deg, W_0 = 0.001eVs$, strong stochastic perturbation $d = 0.01$.

the noise such as to have the equilibrium value of $< x^2 >$ for the harmonic oscillator normalized to 1. These parameters have been chosen in a way to make the effect of the periodic kick force easily visible. In the SDE simulation we took 500 samples for the averaging over the noise process.We made calculations with the corresponding transfer map for the same two systems, using 1000 samples for the averaging process, and compared the mapping results with the SDE results.In figure 1 we show the second moment of x, which measures the bunch size in x-direction. The two lower curves are the results for the unperturbed harmonic oscillator, the two upper curves give the results for the same system being periodically kicked by the beam-beam force. We suppressed the errorbars in this plot for clarity. In both cases, unperturbed (= noisy harmonic oscillator, linear system, $f(x,s) = 0$) and perturbed (= noisy HO plus beam-beam kick force), an equilibrium value is reached. The good agreement of the curves is obvious, although we had "continuous" damping and noise all around the ring in the continuous (within stepwidth h) SDE algorithm, compared to the mapping scheme with already integrated noise and damping applied once, at the interaction point, where the kick is also invoked. Apart from that, the SDE simulation was much more CPU-time consuming.
The calculations were performed on an HP730/9000.

b. Double Rf System with Stochastic Excitation

Noise in the accelerating facilities can lead to significant emittance growth and limit the lifetime of a bunched proton beam [3]. Here we study the influence of phase noise in a double rf system, which is an example of a nonlinear undamped system perturbed by fluctuating forces. J. Wei investigated a double rf system in combination with stochastic cooling [4]. Let ϕ describe the phase deviation of a circulating particle relative to the synchronous particle and $W = \frac{\Delta E}{\omega_{rf}}$ its canonically conjugated variable, the relative deviation in energy from that reference particle. The corresponding differential equation reads:

$$\ddot{\phi} = -C_W C_\phi \sin(\phi) + \frac{1}{m} C_W C_\phi \sin(m\phi) + 2C_W \sqrt{d}\xi(t).$$

where $C_W = \frac{h^2\omega_0^2\eta}{2E\beta^2}, C_\phi = \frac{qe\hat{V}}{\pi h}$, qe=electric charge of the particle, h= harmonic number, γ_t= transition energy, $\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}$, ω_0=revolution frequency of the synchronous particle, βc=velocity of the synchronous particle, $E = m_0 c^2 \gamma$ = synchronous energy, $\sqrt{d}$= scaling diffusion parameter for the noise term, $\xi(t)$= noise process .
For the action variable $J = \oint W d\phi$, which is related to the emittance via $\epsilon = \frac{1}{\pi}J$, we get the approximation [4],[2]: $< J >= const < (C_W W^2 + \frac{1}{2}C_\phi[(1-\cos(\phi)) - \frac{1}{4}(1-\cos(2\phi))])^{\frac{3}{4}} >$, which was used to simulate the growth of the action variable. The parameters used were: $q = 1$, $\hat{V} = 60kV$, $h = 1100$, $m = 2$, $\eta = 5.75 \cdot 10^{-4}$, $\omega_0 = 47kHz \cdot 2\pi$, $\beta = 1.0$, $E = 40GeV$.
Another way to study the synchrotron dynamics is to calculate the FPE in the action variable. Using perturbation theory techniques [3] we get the Fokker-Planck equation:

$$\frac{\partial \rho}{\partial t} = \frac{3}{8} d 2 C_W const^{\frac{4}{3}} \frac{\partial}{\partial J}(J^{\frac{2}{3}} \frac{\partial}{\partial J}\rho(J)).$$

and can derive the SDE in the action variable J [2]:

$$\dot{J} = \frac{1}{3} B J^{-\frac{1}{3}} + \sqrt{2B} J^{\frac{1}{3}} \xi(t),$$

with $B = \frac{3}{8} d 2 C_W const^{\frac{4}{3}}$. Figure 2 shows the results for the emittance growth $< J(t) >$. In this plot we compare the simulations of the ϕ-SDE with those for the J-SDE (500 particles). Both curves agree very well, even on long time scales.

III. COMPUTING DENSITY FUNCTIONS VIA STOCHASTIC MAPPINGS

A quantity of fundamental significance for the description of particle motion in a storage ring is the density function and its evolution in time. It holds the information about beam sizes and lifetimes. In general the density is obtained via tracking many particles over a large number of turns (a few damping times). In what follows we describe a discrete model for calculating the phase space density function $\rho(x,p,t)$ of an electron storage ring in the presence of damping, external noise and nonlinearities

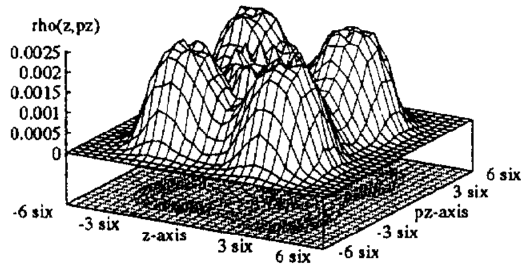

Figure 3: $\rho(z,p_z)$ 1852 turns=τ_d,$Q = 5.24, \xi = 0.029, \alpha = 5.4 \cdot 10^{-4}$, 30×30-grid, absorbing boundaries at 6σ.

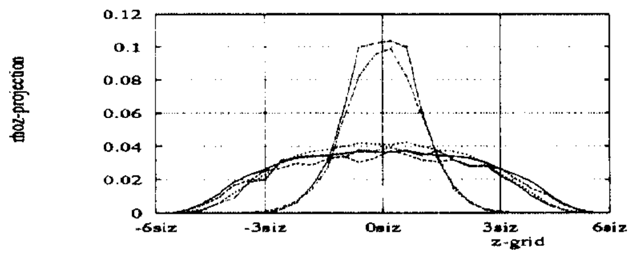

Figure 4: $\rho(z,p_z)$, z-proj., mapping alg.(200p/bin) and tracking (10p/bin), after $\frac{1}{4}\tau_d$, $\frac{1}{2}\tau_d$, $20\tau_d$, $Q = 5.14$.

(here: beam-beam force). The idea of the presented algorithm mainly goes back to A. Gerasimov [5]. It consists of the construction of a time propagator (two-time transition probability) for a discrete Markov process, see also [6].

A. The Time Propagator Matrix

We consider a one-dimensional beam-beam model for the interaction of two round beams. The transformation for one turn consists of the kick at the IP and of the linear part, given by the transfer matrices. The phase space is partitioned into discrete states i ($n \times n$-grid), where a particular state is identified with a position on the grid. For a transition probability having the Markov property, the following holds: $P(x_{n+1} = j|x_n = i; h) = P(x_{n+1} = j|x_n = i) = p_{ij}(n)$. $P(x_{n+1} = j|x_n = i; h)$ is the probability of being in state j at time $n+1$, after having been in state i at time n and having the "history" h. For the two-time transition probability this yields: $p_{ik}^{(2)} = \sum_{j \in I} p_{ij} p_{jk}$. One arranges the probabilities for all possible transitions between the different states as a matrix $A_{ij} = p_{ij}$ (stochastic matrix). Tracking many particles for a certain number of turns, we compute A as the matrix of the relative frequencies of transitions between the different bins of the phase space grid. The number of turns in the tracking can be chosen in different ways, for example such as to have still enough particles in the tails of the distribution. In general one takes one half of the damping time. After we have evaluated the time propagator matrix A, we apply it successively to the initial density ρ_0 and simulate the time evolution of ρ : $\quad \rho_1 = A\rho_0, \qquad \rho_n = A^n \rho_0 = A\rho_{n-1}$.

In figure 3 we see a calculation of the vertical density after one damping time, starting from a homogeneous initial density.

Figure 4 shows a comparison between direct tracking (10 particles/bin) and the mapping algorithm (200 p/bin). The agreement is very good, although after 40000 turns the "mapped density" curve is slightly below the "direct one". The mapping needed about two orders of magnitude less CPU-time.

B. "Macrostates" and Higher-Dimensional Systems

Instead of calculating the transition probability operator for every two gridpoints (or "microstates"), we now search for a partition of the phase space into larger structures ("macrostates") in order to reduce the computing effort and handle higher-dimensional systems. We then compute the transition matrix for these larger units. For two-dimensional systems, it is necessary to find a partition of an $n \times n \times n \times n$-phase space grid. The iteration time parameters therefore have to be chosen in such a way as to get a sufficient small number of macrostates to keep the matrices treatable, but nevertheless the macrostates should still represent the structure of the phase space.

IV. CONCLUSIONS

We introduced several numerical integration algorithms for SDEs and applied them to examples in accelerator physics. Although the results are good, the methods are very CPU time consuming. By making a combined analytical and numerical analysis of a nonlinear rf system with phase noise we could show that simulations with SDEs and analytical perturbation theory were in excellent agreement.

In the second part of the presented work we have introduced and tested an algorithm to investigate the motion of ultrarelativistic charged particles under the influence of damping, noise, and certain non-linear forces. Via a stochastic mapping we calculated a time propagator and computed a "numerical Markov chain" for the probability density on the phase space. By using this method one gets a very good impression of the time evolution of the density function. Resonance structures on the phase space can be made easily visible. Besides, the algorithm is by 2-3 orders of magnitude faster than direct tracking methods.

V. REFERENCES

[1] Honerkamp,J., Stochastische dynamische Systeme, VCH, Weinheim, (1990)

[2] Pauluhn,A., Stochastic Beam Dynamics with Special Applications to a Double Rf System, DESY HERA 93-02, (1993)

[3] Krinsky,S.,Wang,J.M., Bunch Diffusion due to RF Noise, Part. Acc., 12, 1982

[4] Wei,J., Stochastic Cooling with a Double RF System, BNL-46792, AD/RHIC-105, Informal Report, (1991)

[5] Gerasimov,A., Private communication

[6] Pauluhn,A., Gerasimov,A., Mais,H., A Stochastic Map for the One-Dimensional Beam-Beam Interaction, in the Proc. of the 15th Int. Conf. on High Energy Accelerators, Hamburg, (1992)