Statistics of the Half-Integer Stopband

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Abstract

We consider the statistical nature of the stopband set up in the vicinity of a half-integer tune when a linear lattice is subjected to quadrupole errors distributed according to gaussians. The probability density function of the stopband, treated as a complex number, is found to be a correlated bivariate gaussian in the real and imaginary parts. The mean magnitude of the stopband is calculated in terms of the complete elliptic integral of the first kind, and the conditional probability density of its magnitude is obtained in closed form. A number of limiting conditions are studied. Finally, we estimate the requirements on a correction system for neutralizing the stopband with a given probability of success.

I. INTRODUCTION: PHYSICS

Consider a linear lattice subjected to small quadrupole errors. Fig. 1(a) displays the eigenvalues of the one-dimensional transfer matrix in the complex plane [1]. If the unperturbed tune ν lies close to a half-integer, the eigenvalues $e^{\pm i\mu}$, $\mu = 2\pi\nu$, will lie on either side of the negative real axis, as shown. The influence of the quadrupole errors will be either (i) to rotate the eigenvalues away from the negative real axis, or (ii) toward it. If the effect is the latter, then as the strength of the perturbation increases, the eigenvalues will coincide on the negative real axis at some point. At this stage, further increasing the strength of the perturbation will either (i) move the eigenvalues past each other on the unit circle, or (ii) cause them to move onto the real axis, thereby making the lattice unstable. Lattice configurations corresponding to these two possibilities are displayed in Figs. 1(b) and 1(c), where we assume that the unperturbed lattice has a superperiod of two, and the perturbing quadrupoles, of equal strength, to be separated by a superperiod. In the first case, Fig. 1(b), the eigenvalues would rearrange themselves on the unit circle. In the second case, Fig. 1(c), the eigenvalues would move onto the real axis if the perturbation is sufficiently strong. It is customary to attribute the resulting instability to the *half-integer stopband*. Note that there is no stopband for the perturbed lattice, in the sense that the tune must reach the half-integer before the lattice becomes unstable [3]. One can, however, speak of a stopband for the *unperturbed* lattice, Fig. 2. The tune shift caused by the perturbing

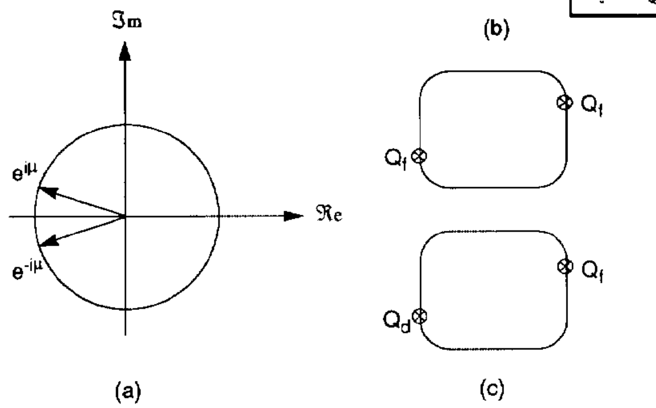

Figure 1: (a) Eigenvalues of one-dimensional transfer matrix near half-integer tune. (b) Stable, and (c) possibly unstable lattice configurations near half-integer with perturbing quadrupoles separated by a superperiod.

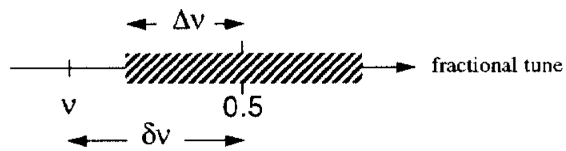

Figure 2: The half-integer stopband as seen by the unperturbed lattice.

quadrupoles in Fig. 1(c) is

$$\text{tune shift} \approx \delta\nu \times \left[1 - \sqrt{1 - \left(\frac{\Delta\nu}{\delta\nu}\right)^2}\right], \qquad (1)$$

where $\Delta\nu$, called the *stopband halfwidth*, is [2]

$$\Delta\nu = \left|\frac{1}{4\pi}\oint ds\, Q(s)\beta(s)\, e^{i(2W+ps/R)}\right|, \qquad (2)$$

and

$$W(s) = \mu(s) - \nu\frac{s}{R}, \quad \mu(s) = \int_0^s \frac{ds'}{\beta(s')}.$$

$Q(s)$, the distribution of quadrupole errors, is the gradient error normalized by the nominal rigidity, p the $2\nu^{\text{th}}$ harmonic, s the longitudinal coordinate, and R the average machine radius. Lattice functions in (2) correspond to the unperturbed lattice. In this view, the unperturbed tune must lie outside the shaded region set up by the perturbing quadrupoles, *i.e.*, $\delta\nu \geq \Delta\nu$, or the tune shift given by (1) becomes complex, and the lattice becomes unstable. Hence, the shaded region about the half-integer is referred to as a stopband.

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*Operated by the Universities Research Association, Inc., for the U. S. Department of Energy under Contract No. DE-AC35-89ER40486.

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II. STATISTICS

The half-integer stopband $\Delta\nu$ is the magnitude of the complex integral in (2), which we denote by $\Delta\nu_c$. The only statistical element which appears in (2) is the factor of $Q(s)$, the distribution of quadrupole errors in the lattice elements. If $\beta(s)$ varies slowly in the lattice elements, we can make the approximation

$$\Delta\nu_c \approx \frac{1}{4\pi}\sum_k Q_k\beta_k\ell_k\, e^{i(2W_k+ps_k/R)}, \tag{3}$$

where the index k runs over the lattice elements, with ℓ_k the length of the k^{th} element. Given that Q_k is a gaussian random variable with zero mean and RMS variation of σ_k, we are required to compute the mean and variance of $\Delta\nu = |\Delta\nu_c|$.

We recast the problem in more abstract form. Let $Z = (X, Y)$ be a complex number made up as

$$X = \sum_k c_k\, x_k, \quad Y = \sum_k d_k\, x_k, \tag{4}$$

where $c_1, \ldots, c_n$, $d_1, \ldots, d_n$ are real constants, and $x_1, \ldots, x_n$ are independent gaussian random variables with

$$< x_k >= 0, \quad < x_k^2 >= \varsigma_k^2. \tag{5}$$

We are required to compute

$$\left\langle \sqrt{X^2+Y^2}\,\right\rangle, \text{ and } \left\langle X^2+Y^2\right\rangle.$$

We state without proof a lemma used to calculate the joint PDF of X and Y:

■ Let $x_1, \ldots, x_n$ be independent random variables with PDF's $p_1, \ldots, p_n$ respectively. Let X, Y be functions of the independent variables, or

$$X = F(x_1, \ldots, x_n), \quad Y = G(x_1, \ldots, x_n).$$

If $P(X, Y)$ denotes the joint PDF of X and Y, then

$$P(X,Y) = \int dx_1\cdots dx_n\, p_1\times\ldots p_n\delta(X-F)\,\delta(Y-G)$$

where δ denotes the Dirac delta function. In other words, the joint PDF of X, Y is the average over $x_1, \ldots, x_n$ of $\delta\,(\,X - F(x_1, \ldots, x_n)\,) \times \delta\,(\,Y - G(x_1, \ldots, x_n)\,)$.

■ The joint PDF of X and Y in (4), works to be

$$\hat{P}(X,Y) = \frac{1}{2\pi\sqrt{\sigma_X^2\,\sigma_Y^2 - <XY>^2}}\times \exp\left[-\frac{X^2\sigma_Y^2+Y^2\sigma_X^2-2XY<XY>}{2(\sigma_X^2\sigma_Y^2-<XY>^2)}\right], \tag{6}$$

where

$$\sigma_X^2 \equiv \langle X^2\rangle - \langle X\rangle^2, \quad \sigma_Y^2 \equiv \langle Y^2\rangle - \langle Y\rangle^2,$$

$$\langle X\rangle = \langle Y\rangle = 0,$$

$$\langle X^2\rangle = \sum_k c_k^2\,\varsigma_k^2, \quad \langle Y^2\rangle = \sum_k d_k^2\,\varsigma_k^2,$$

$$\langle XY\rangle = \sum_k c_k\,d_k\,\varsigma_k^2.$$

From (6) we see that if $\langle XY\rangle\ = 0$, X and Y become uncorrelated gaussian random variables.

■ The average of $\sqrt{X^2+Y^2} \equiv r$ works out to be

$$\langle r\rangle\ = \left(\frac{\pi}{2}\right)^{\frac{1}{2}}\left[\sigma_X^2\,\sigma_Y^2 - \langle XY\rangle^2\right]^{\frac{1}{4}} P_{\frac{1}{2}}(\zeta), \tag{7}$$

where

$$\zeta = \frac{\sigma_X^2+\sigma_Y^2}{2\sqrt{\sigma_X^2\,\sigma_Y^2 - \langle XY\rangle^2}}. \tag{8}$$

$P_{\frac{1}{2}}$ is the Legendre function of index half[4]. It is easily established that ζ, the argument of $P_{\frac{1}{2}}$, is greater than or equal to 1. For values of $\zeta \geq 1$, $P_{\frac{1}{2}}$ can be expressed as

$$P_{\frac{1}{2}}(\zeta) = \frac{2}{\pi}\left[\zeta+\sqrt{\zeta^2-1}\right]^{\frac{1}{2}} E(m),$$

where

$$m = \left[\frac{2\sqrt{\zeta^2-1}}{\zeta+\sqrt{\zeta^2-1}}\right]^{\frac{1}{2}}, \text{ and } E(m) = \int_0^{\frac{\pi}{2}} d\theta\,\sqrt{1-m\sin^2\theta}$$

is a complete elliptic integral of the first kind [4]. Combining these results, we obtain

$$\langle r\rangle\ = \sqrt{\sigma_X^2+\sigma_Y^2}\ f(\zeta), \tag{9}$$

where

$$f(\zeta) = \left[\frac{1}{\pi}\left(1+\sqrt{1-\zeta^{-2}}\right)\right]^{\frac{1}{2}} E(m).$$

The average of r^2 can be obtained directly from (4) and (5)

$$\left\langle r^2\right\rangle = \left\langle X^2+Y^2\right\rangle = \sigma_X^2+\sigma_Y^2 \equiv \sigma^2. \tag{10}$$

■ From the properties [4] of $E(m)$ we obtain

$$0.8\,\sigma \leq \langle r\rangle \leq 0.9\,\sigma. \tag{11}$$

A simple and robust approximation is thus obtained

$$\langle r\rangle \approx \sigma. \tag{12}$$

Also, from (11) we have

$$0.19\,\sigma^2 \leq \text{var}(r) \leq 0.36\,\sigma^2.$$

■ The significance of the variable ζ, defined in (8), requires comment. The smallest value ζ can have is unity. It obtains when $< XY > = 0$ and $\sigma_X = \sigma_Y$, *i.e.*, when X and Y are uncorrelated and have the same variance. However, a value of ζ larger than unity does not necessarily signify

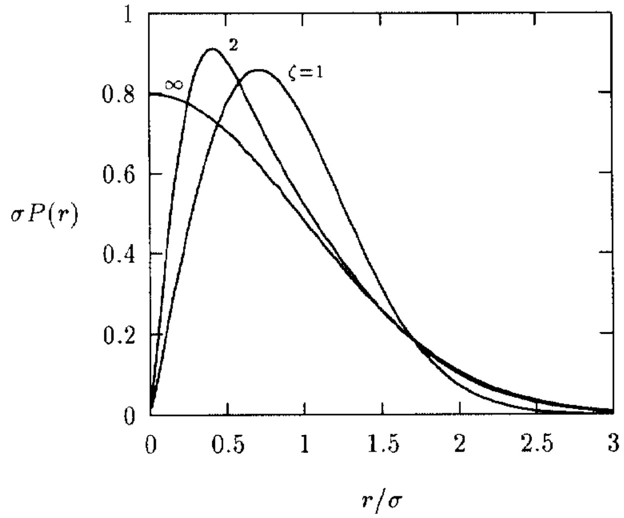

Figure 3. Conditional probability density function of the magnitude of the stopband, for $\zeta = 1$, $\zeta = 2$, and $\zeta \to \infty$.

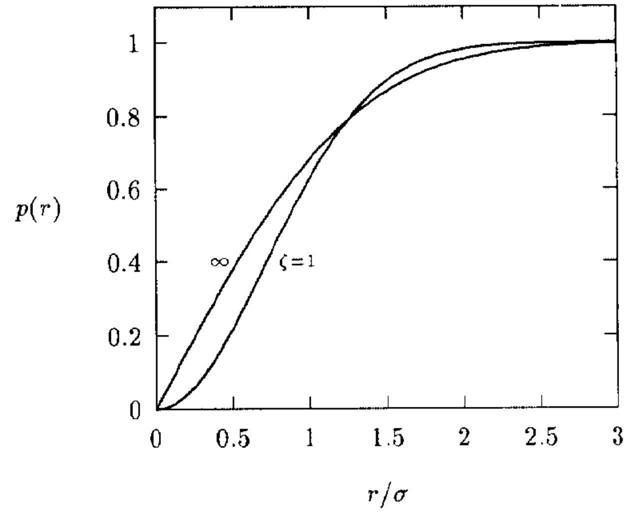

Figure 4. Probability of finding r between 0 and some prescribed limit, for $\zeta = 1$ and $\zeta \to \infty$. For intermediate values of ζ, the curve will lie within the envelope defined above.

an increasing degree of correlation. This can be seen from (6), which requires only that $< XY > = 0$ for X and Y to be uncorrelated variables, in which case

$$\zeta \to \frac{1}{2}\left(\frac{\sigma_X}{\sigma_Y} + \frac{\sigma_Y}{\sigma_X}\right).$$

Accordingly, if $\sigma_X \neq \sigma_Y$, ζ will be greater than unity, and will become large if $\sigma_X \ll \sigma_Y$, or vice versa. Another way for ζ to become larger than unity is when X and Y are correlated, which requires $< XY > \neq 0$. The degree of correlation between X and Y is measured by how close the value of $| < XY > |$ comes to $\sigma_X\, \sigma_Y$. For example, if the constants $c_1, \ldots, c_n$ and $d_1, \ldots, d_n$ in (4) are identical, then X is identical to Y, which represents perfect correlation. In this case one obtains $\sigma_X\, \sigma_Y = < XY >$. This is sufficient to make the value of $\zeta \to \infty$, as in the case of uncorrelated variables with grossly different variances.

■ The conditional probability density function $P(r)$ of the magnitude of the stopband r is

$$P(r) = 2\,\frac{\zeta r}{\sigma^2}\exp\left(-\frac{\zeta^2 r^2}{\sigma^2}\right) I_0\left(\sqrt{1-\zeta^{-2}}\,\frac{\zeta^2 r^2}{\sigma^2}\right). \quad (13)$$

Fig. 3 displays $\sigma P(r)$ as a function of r for three different values of ζ.

■ The limiting cases $\zeta = 1$ and $\zeta \to \infty$ can be worked out explicitly, and provide some insight into the behavior of $P(r)$ as a function of ζ. In the first case, $\zeta = 1$, one obtains from (13)

$$P(r) = 2\,\frac{r}{\sigma^2}\exp\left(-\frac{r^2}{\sigma^2}\right).$$

This is precisely the conditional PDF in the radial variable for a bivariate gaussian with equal variances. In the second limiting case, $\zeta \to \infty$, we use the asymptotic form [4] of $I_0(z)$ to obtain

$$P(r) \sim \sqrt{\frac{2}{\pi\sigma^2}}\,\exp\left(-\frac{r^2}{2\sigma^2}\right).$$

This is a "half" gaussian, as can be seen in Fig. 3. Since $\zeta \to \infty$ represents (i) the case of perfectly correlated variables, or (ii) the case when the individual variances of X and Y are grossly different, we expect r to be determined by a single variable. In this case, the resulting distribution should remain gaussian, except that it must be positive. Hence, a half-gaussian. For intermediate values of ζ, the peak of the PDF moves closer to the origin, and the tail decays more rapidly, *i.e.*, it approaches the half-gaussian case.

■ The probability

$$p(r) = \int_0^r d\bar{r}\, P(\bar{r})$$

of finding r between 0 and an arbitrary multiple of σ up to 3σ is shown in Fig. 4 for the limiting values of $\zeta = 1$ and $\zeta \to \infty$. Probability curves for intermediate values of ζ will lie within the envelope defined by the limiting cases.

■ Finally, we illustrate the use of these results. Say we would like to design a correction system for the stopband with a 0.95 probability of being sufficiently strong. Fig. 4 tells us that the correction system must have a driving term (2) equal to 1.7σ for $\zeta = 1$, and 2.0σ for $\zeta = \infty$. σ and ζ would have to be computed from the correspondence between (3) and (4), and knowledge of the machine lattice and its error distribution.

III. REFERENCES

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