Higher order tune derivatives due to low-β insertions.

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Abstract

In an electron machine the adjustment of the first order derivatives of the tunes with respect to momentum is important to counteract the dipole mode head-tail instability. In the case where low-β insertions are included in the lattice, this first order correction is not sufficient because important higher order tune derivatives appear. The associated strong quadratic variation of the tunes with momentum, results in a linear betatron instability for a small number of standard deviations in energy. This makes the life-time unacceptably small. The origin of tune derivatives of order larger than one is explained. The principle of their correction is recalled.

I. Introduction

The chromaticity problem is approached here as a particular case of a general treatment of gradient perturbations which had been developed for optimizing an imperfect matching [1]. This treatment, which deals with a one-degree of freedom motion, will first be recalled. Then its application to the higher order chromaticity due to a low-β insertion will be done. The compensation with sextupole families will then be shortly examined. These matters had been developed in a course given by the author [2]. The aim of this paper is to explain more clearly the formula for Q" which is the key point of the treatment and to correct some minor mistakes.

II. Global estimation of a gradient perturbation

We consider a perfect machine at the end of which the betatron-functions have the values β and α. By definition the betatron functions have also the values β and α at the beginning of the machine.

We introduce in this machine a certain gradient perturbation. The effect of this perturbation can be computed exactly by means of the transforms of β and α through the perturbed machine, which are β^t and α^t, and the associated phase advance μ^t defined by :

$$\mu^t = \int_0^C \frac{ds}{\beta^t} \tag{1}$$

These quantities are indeed enough to obtain the perturbed one turn matrix [3], the elements of which are:

$$m_{11} = \sqrt{\frac{\beta^t}{\beta}}(\cos\mu^t + \alpha\sin\mu^t)$$

$$m_{12} = \sqrt{\beta\beta^t}\sin\mu^t$$

$$m_{21} = \frac{1}{\sqrt{\beta\beta^t}}\left((1+\alpha\alpha^t)\sin\mu^t + (\alpha^t-\alpha)\cos\mu^t\right)$$

$$m_{22} = \sqrt{\frac{\beta}{\beta^t}}(\cos\mu^t - \alpha_t\sin\mu^t)$$

It is important to note that β^t and α^t are not true Twiss-functions : they have the same meaning as Twiss-functions in a transfer line. The true β-function β^* at the end of the perturbed machine can be obtained from the second element of the first line of this matrix :

$$\beta^* = \sqrt{\beta\beta^t}\sin\mu^t/\sin\mu^*$$

The new tune μ^* can be computed from the trace of the perturbed matrix :

$$2\cos\mu^* = \left(\sqrt{\frac{\beta^t}{\beta}} + \sqrt{\frac{\beta}{\beta^t}}\right)\cos\mu^t + \left(\alpha\sqrt{\frac{\beta^t}{\beta}} - \alpha^t\sqrt{\frac{\beta}{\beta^t}}\right) sin\mu^t \tag{2}$$

Putting :

$$\theta = \arctan\frac{\alpha\sqrt{\frac{\beta^t}{\beta}} - \alpha^t\sqrt{\frac{\beta}{\beta^t}}}{\sqrt{\frac{\beta^t}{\beta}} + \sqrt{\frac{\beta}{\beta^t}}} = \frac{\beta^t\alpha - \alpha^t\beta}{\beta+\beta^t}$$

We can transform equation (2) into :

$$\cos\mu^* = \cos(\mu^t+\theta)\times$$
$$\sqrt{1+\frac{1}{4}\left(\sqrt{\frac{\beta^t}{\beta}} - \sqrt{\frac{\beta}{\beta^t}}\right)^2 + \frac{1}{4}\left(\alpha\sqrt{\frac{\beta^t}{\beta}} - \alpha^t\sqrt{\frac{\beta}{\beta^t}}\right)^2} \tag{3}$$

In order to obtain this expression, there is a trick consisting of adding 4 to the sum of the squares of the coefficients of the trigonometric functions in equation (2), so that the sign plus in the first one can be changed to minus. As the term under the square root is always larger than 1, there are values of $\mu^t+\theta$ for which $\cos\mu^*$ is larger than 1, even if

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the unperturbed $\cos\mu$ is smaller than 1 : the gradient perturbation has opened 'gradient stopbands'. An illustration of this effect can be found in ref [1].

It is worth noting that the expression under the square root can be used as a measure of mismatch when trying to match an insertion. In the case of an imperfect matching, minimizing this expression guarantees that the stopbands associated with the mismatch have the minimum width.

III. Chromatic perturbation

We expand μ^t and μ^* in power series of the relative momentum deviation δ. μ^* is then $2\pi Q(\delta)$, Q being the tune of the machine, which is a function of the momentum deviation.

$$\mu^t = \mu + \mu^{t\prime}\delta + \frac{1}{2}\mu^{t\prime\prime}\delta^2 + ...$$

$$\mu^* = \mu + \mu'\delta + \frac{1}{2}\mu''\delta^2 + ...$$

We expand also β^t and α^t :

$$\beta^t = \beta + \beta'\delta + ..., \ \ \alpha^t = \alpha + \alpha'\delta + ...$$

β and α being the on-momentum values. α' and β' are not the derivatives of α and β with respect to momentum, but the derivative of the transforms of the on-momentum functions through the machine. The computation of this β' can be found in [4]. For instance the contribution to this derivative of a thin quadrupole of length l and normalized gradient k is :

$$\frac{\beta'}{\beta} = -kl\beta_0 \sin 2[\mu - \mu_0] \tag{4}$$

where the unlabeled optics parameters refer to the point of longitudinal coordinate s where the derivative is computed and the quantities labeled 0 refer to the quadrupole location. Taking the derivative of 4 with respect to s :

$$\alpha\frac{\beta'}{\beta} - \alpha' = kl\beta_0 \cos 2[\mu - \mu_0] \tag{5}$$

Now we carry on with the identification of the terms with the same power of δin the LHS and RHS of equation (3) after expansions in power of δ. The terms in δ give:

$$\mu' = \mu^{t\prime} + \frac{1}{2\beta}[\alpha\beta' - \beta\alpha']$$

μ' is the first derivative of the tune of the perturbed machine multiplied by 2π , $\mu^{t\prime}$ is obtained from 1.

Then, identifying the terms in δ^2, we obtain :

$$\mu'' = \mu^{t\prime\prime} + \left[\frac{\alpha\beta'}{\beta} - \alpha'\right]' + \mu^{t\prime}\left[\frac{\alpha\beta'}{\beta} - \alpha'\right]\cot\mu$$

$$-\frac{1}{4}\cot\mu\left[\left(\frac{\beta'}{\beta}\right)^2 + \left(\frac{\alpha\beta'}{\beta} - \alpha'\right)^2\right] \tag{6}$$

In this expression the last term is much larger than the other ones as long as the first derivative $\mu^{t\prime}$ is some units. It describes the effect of the first order off-momentum mismatch of the β-function due to the low-β insertion. The numerical support of these statements is given below.

Identifying the terms in δ^3 leads to a similar result. The large term is still there, as well as its derivative with respect to momentum.

IV. Q" due to a low-β insertion

We consider the case of a machine composed of $\mathcal{N}_s$ superperiods with one symmetric low-β insertion per superperiod. All machine quadrupoles contribute to the chromatic effects but there is at least a strong one, close to the crossing point, which has a dominant effect on the second order tune derivative. In order to give an idea of the order of magnitude of this effect, we can consider the case of LEP under physics conditions, in the vertical plane. The β-value at the interaction point β^* is 5cm. The closest quadrupole is at 3.7m, it has a length of 2m and a strength k of $0.164m^{-2}$. The β-value at the quadrupole centre is about 400m, and the expression $kl\beta_0$ has a value of about 130. For the other lattice quadrupoles of length 2m, k is always below 0.03 and the β-value below 140m, resulting in $kl\beta_0$ smaller than 8. In equation 6 all contributions add-up linearly with phase terms as given by formulae 4 and 5, so that the effect of the low-β quadrupole dominates.

Keeping only the effect of the off-momentum mismatch due to two low-β quadrupoles in phase (they are π apart) per insertion, formula 6 takes the form :

$$Q'' \simeq -\frac{\mathcal{N}_s}{2\pi}(kl\beta)^2 \cot\frac{2\pi Q}{\mathcal{N}_s} \tag{7}$$

Q being the tune of the machine, Q'' its second derivative with respect to momentum and $\mathcal{N}_s$ the number of superperiods.

The cotangent of the tune per superperiod is an important factor. If it is close to a half integer, the cotangent becomes very large. This is precisely a condition favorable for the beam-beam effect, because of the associated reduction of the beam size at the crossing point.

With the above mentioned LEP parameters of the low-β quadrupoles, Q'' given by formula 7 is 3.3×10^4. A tune shift of -0.2 is enough to produce a betatron instability since the fractional part of the vertical tune is 0.2. This tune shift is obtained with a momentum deviation of 3.5×10^{-3} with this value of Q''. The actual variation of the tunes with momentum of LEP with two sextupole families, for the physics optics, is shown on fig 1. We observe that the betatron instability occurs indeed at about 3×10^{-3} in the vertical plane. This shows the dominant effect of the off-momentum mismatch on the second order tune derivative for this plane. In the horizontal plane the variation of the tune with momentum has a much smaller curvature. This is because the $kl\beta$ value is smaller than that for the vertical plane by a factor of about five. A similar situation occurs in the vertical plane for the injection optics where $\beta_y{}^*$ is reduced by a factor three, which reduces the contribution to Q'' by one order of magnitude.

If the tune per superperiod is close to a quarter integer modulo one half, the cotangent becomes small and tunes satisfying this condition avoid taking care for the compensation. This is for instance what has been done to test LEP with 90° cells. Choosing the tunes:

$$Q_h = 91.30 \quad Q_v = 97.20$$

makes it possible correct the chromaticity with two sextupole families for a β^* of 5cm [5], one order of magnitude being gained on cot $\frac{2\pi Q}{N_s}$.

The machine parameters making $kl\beta$ large are mainly β^* and the distance L between the crossing point and the centre of the low-β quadrupole. $kl\beta$ is determined by the necessity of changing the sign of the derivative of β after the low-β quadrupole. For a thin lens model, we have :

$$kl\beta = -2\alpha = 2L/\beta^*$$

We can check for the LEP parameters given above that this expression gives 190, which is quite close to the actual value of 130. Putting this expression in 7, we see that Q'' scales with $(L/\beta^*)^2$.

V. Compensation of the off-momentum mismatch

An obvious solution is to use sextupoles to match the first derivative of the β-function. As there are already such elements to adjust the first derivative of the tunes, the best procedure is to split them into families in order to make "off-momentum cells" which match the first derivative of the tune with respect to momentum. A variety of such arrangements has been tried. A review can be found in ref [2]. The experience shows that the best procedure consists in splitting the sextupoles into families as regular as possible with a phase advance per cell close to a simple fraction of π. This phase constraint guarantees both that the correction is possible, as two sextupoles separated by a π phase advance act in phase for the correction of β' (see equations 4 and 5), and that the non-linear transverse oscillations have the least detrimental effect.

VI. Conclusion

Increasing a lepton storage ring luminosity by decreasing the value of the β-function at the crossing point is limited by the non-linear chromaticity produced by the low-β insertion. The essential part of this effect is due to the second tune derivative with respect to momentum. When β^* is decreased below a certain threshold, this contribution to the non-linear chromaticity has to be compensated by splitting the sextupoles into families.

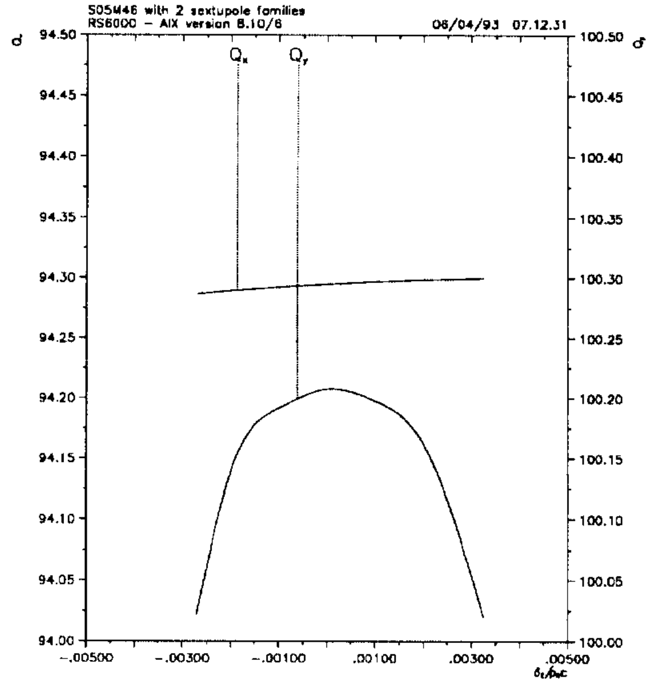

Figure 1: Variation of the tunes with relative momentum deviation for the LEP optics used in 1992. Qx=94.3, Qy=100.2 on the central orbit. There are four superperiods with one low-β insertion per superperiod. The first derivatives of the tunes have been set to about 2.5 with two sextupole families

VII. References

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