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Symmetric Integrable-Polynomial Factorization for Symplectic One-turn-Map Tracking

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Abstract

It was found that any homogeneous polynomial can be written as a sum of integrable polynomials of the same degree by which Lie transformations can be evaluated exactly. By utilizing symplectic integrators, an integrable-polynomial factorization is developed to convert a symplectic map in the form of Dragt-Finn factorization into a product of Lie transformations associated with integrable polynomials. A small number of factorization bases of integrable polynomials enables one to use high-order symplectic integrators so that the high-order spurious terms can be greatly suppressed. A symplectic map can thus be evaluated with desired accuracy.

I. INTRODUCTION

In large storage rings, high-intensity beams are required to circulate for many hours in the presence of nonlinear perturbations of multipole errors in magnets. Extensive computer simulations are thus necessary to investigate the long-term stability of beams. The conventional approach in which trajectories of particles are followed element by element through accelerator structures is, however, very slow in these situations. A substantial computational as well as conceptual simplification is to study the stability of particles by using one-turn maps.

While finding a closed analytical form of a one-turn map is impossible for a large-storage ring with thousands of elements, a truncated Taylor expansion of one-turn map—the Taylor map—can be easily obtained. Even though some successes have been reported using the Taylor maps, the truncation inevitably violates the symplectic nature of systems and consequently leads to spurious effects if the maps are used to study the long-term stability [1]. A reliable long-term tracking study with the Taylor map is therefore possible only if its nonsymplecticity effect can be eliminated without much reduction in the tracking speed.

In order to eliminate the nonsymplecticity, the Taylor map is usually converted into Lie transformations with Dragt-Finn factorization [2]. A map in the form of Lie transformations is guaranteed to be symplectic, but generally cannot be used for tracking directly because evaluating a nonlinear map in such a form is equivalent to solving nonlinear Hamiltonian systems which cannot be done in general. Several methods, such as jolt factorization [3] and monomial factorization [4], have been proposed to deal with this difficulty by converting the Lie transformation from its general the form into special forms that can be evaluated directly. While these methods seem promising, their applications lead to considerable theoretical and computational complexities, chief of which is unpredictability of high-order spurious terms that may lead to a less than accurate evaluation of the map.

Since a general Lie transformation corresponds to a nonintegrable system that cannot be evaluated exactly, the challenge here is how to evaluate a Lie transformation approximately without violating the symplecticity and with a controllable accuracy. One way is to divide the nonintegrable system into subsystems that are integrable individually. The set of subsystems of minimum number is the most promising one to serve as the zeroth-order approximation because it would generate less high-order error and be a better starting point for higher-order treatments. For Lie transformations associated with homogeneous polynomials, we have shown [5] that any polynomial can be written as a sum of integrable polynomials by which Lie transformations can be evaluated exactly. Since the number of integrable polynomials can be much smaller than the number of monomials, a factorization based on the integrable polynomials will have many fewer terms so that a higher order factorization becomes practical. In order to achieve an optimization between a desired accuracy and a fast tracking speed, we have proposed a factorization on the integrable polynomials with symplectic integrators [5]. The advantage of the factorization with symplectic integrators is the suppression of high-order spurious terms to a desired accuracy [6–8].

II. INTEGRABLE POLYNOMIAL IN LIE TRANSFORMATION

A polynomial in $\vec{z}$ is called an integrable polynomial if

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its associated Hamiltonian system is integrable, i.e., its associated Lie transformation can be evaluated exactly. Let $\{g_i^{(k)}|k=1,2,...,N_g\}$ denote a set of integrable polynomials of degree i. In what follows, we shall show that any polynomial in $\vec{z}$ can be expressed as a sum of integrable polynomials of the same degree, i.e.,

$$f_i(\vec{z}) = \sum_{(\sum \sigma_j)=i} a(\vec{\sigma}) z_1^{\sigma_1} p_1^{\sigma_2} z_2^{\sigma_3} p_2^{\sigma_4} z_3^{\sigma_5} p_3^{\sigma_6} = \sum_{k=1}^{N_g} g_i^{(k)}, \quad (1)$$

where f_i is any homogeneous polynomial of degree i in phase-space vector $\vec{z}=(z_1,p_1,z_2,p_2,z_3,p_3)$ and $a(\vec{\sigma})$s are constant coefficients. After factorizing it as a product of Lie transformations associated with integrable polynomials, $\exp(:f_i:)\,\vec{z}$ can be therefore evaluated directly. Since the minimum number of integrable polynomials N_g is much smaller than the number of monomials, the accuracy of factorization with $\{g_i^{(k)}\}$ as bases can be carried to a desired order with the use of symplectic integrators.

Homogeneous polynomials of degree 3 in 6-variables consist of 56 monomials, which can be grouped under 8 integrable polynomials of degree 3, $\{g_3^{(n)}|n=1,2,...,8\}$:

$$\begin{aligned} g_3^{(1)} &= c_1^{(1)} z_1^3 + c_2^{(1)} z_1^2 p_1 + c_3^{(1)} z_2^3 \\ &\quad + c_4^{(1)} z_2^2 p_2 + c_5^{(1)} z_3^3 + c_6^{(1)} z_3^2 p_3, && (2)\\ g_3^{(2)} &= c_1^{(2)} p_1^3 + c_2^{(2)} p_1^2 z_1 + c_3^{(2)} p_2^3 \\ &\quad + c_4^{(2)} p_2^2 z_2 + c_5^{(2)} p_3^3 + c_6^{(2)} p_3^2 z_3, && (3)\\ g_3^{(2+i)} &= z_i h_2^{(2+i)}(z_j,p_j,z_k,p_k), && (4)\\ g_3^{(5+i)} &= p_i h_2^{(5+i)}(z_j,p_j,z_k,p_k), && (5) \end{aligned}$$

where (i,j,k) goes over all cyclic permutations of $(1,2,3)$, $h_2^{(n)}$s are homogeneous polynomials of degree 2 in 4-variables, and $c_m^{(n)}$ is the coefficient of the corresponding monomial in f_3. It should be noted that the decomposition of f_i into integrable polynomials is not unique. $g_3^{(1)}$ and $g_3^{(2)}$ can be further combined into a single integrable polynomial since the Hamiltonian system with $H=-(g_3^{(1)}+g_3^{(2)})$ is integrable. We chose two separate integrable polynomials instead of the combined one because the solution for the later cannot be written in a closed form and directly used in tracking.

The Lie transformations associated with integrable polynomials can be converted into simple iterations [5]:

$$e^{:g_3^{(1)}:} z_i = \frac{z_i}{1+c_{2i}^{(1)} z_i}, \quad (6)$$

$$e^{:g_3^{(1)}:} p_i = -\frac{(c_{2i-1}^{(1)} z_i + c_{2i}^{(1)} p_i)(c_{2i}^{(1)} z_i + 1)^3 + c_{2i-1}^{(1)} z_i}{c_{2i}^{(1)}(1+c_{2i}^{(1)} z_i)}, \quad (7)$$

$$e^{:g_3^{(2)}:} z_i = \frac{(c_{2i-1}^{(2)} p_i + c_{2i}^{(2)} p_i)(c_{2i}^{(2)} p_i - 1)^3 - c_{2i-1}^{(2)} p_i}{c_{2i}^{(2)}(1-c_{2i}^{(2)} p_i)}, \quad (8)$$

$$\begin{aligned} e^{:g_3^{(2)}:} p_i &= \frac{p_i}{1-c_{2i}^{(2)} p_i}, && (9)\\ e^{:g_3^{(2+i)}:} z_i &= z_i, && (10)\\ e^{:g_3^{(2+i)}:} p_i &= p_i + h_2^{(2+i)}(z_j,p_j,z_k,p_k), && (11)\\ e^{:g_3^{(5+i)}:} z_i &= z_i - h_2^{(5+i)}(z_j,p_j,z_k,p_k), && (12)\\ e^{:g_3^{(5+i)}:} p_i &= p_i, && (13)\\ e^{:g_3^{(2+i)}:} \vec{r} &= U_{2+i}^{-1} \exp(z_i \Lambda_{2+i})\, U_{2+i} \vec{r}, && (14)\\ e^{:g_3^{(5+i)}:} \vec{r} &= U_{5+i}^{-1} \exp(p_i \Lambda_{5+i})\, U_{2+i} \vec{r}, && (15) \end{aligned}$$

where (i,j,k) goes over all cyclic permutations of $(1,2,3)$.

$$\vec{r} = (z_j \quad p_j \quad z_k \quad p_k)^T,$$

$$\frac{\partial}{\partial \vec{r}} = \left(\frac{\partial}{\partial z_j} \quad \frac{\partial}{\partial p_j} \quad \frac{\partial}{\partial z_k} \quad \frac{\partial}{\partial p_k}\right)^T,$$

$$\Lambda_n = U_n \left[\Gamma \frac{\partial}{\partial \vec{r}} \left(\frac{\partial}{\partial \vec{r}}\right)^T h_l^{(n)}\right] U_n^{-1} = \begin{pmatrix} \lambda_1^{(n)} & 0 & 0 & 0 \\ 0 & \lambda_2^{(n)} & 0 & 0 \\ 0 & 0 & \lambda_3^{(n)} & 0 \\ 0 & 0 & 0 & \lambda_4^{(n)} \end{pmatrix}, \quad (16)$$

where superscript T denotes the transpose and Γ is a 4-dimensional antisymmetric matrix:

$$\Gamma = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (17)$$

Similarly, 126, 252, and 462 monomials of homogeneous polynomials of degree 4, 5, and 6 in 6-variables can be grouped into 20, 42, and 79 integrable polynomials of degree 4, 5, and 6, respectively [5].

III. SYMMETRIC INTEGRABLE-POLYNOMIAL FACTORIZATION

With integrable polynomials, a symplectic map in the form of the Dragt-Finn factorization can be rewritten as

$$\vec{U}_s(\vec{z}) = R \prod_{i=3} \exp\left(\sum_{n=1}^{N_i} :g_i^{(n)}:\right) \vec{z}, \quad (18)$$

where R denotes the linear transformation and N_i is the number of integrable polynomials of degree i. By means of Campbell-Baker-Hausdorff (CBH) formula [2], one can, in principle, convert the Lie transformation associated with a sum of integrable polynomials into a product of Lie transformations associated with integrable polynomials. Since those integrable Lie transformations of the same order are, in general, not commutable, such nonsymmetric separation will cause spurious errors on the next and higher orders. We therefore propose that symplectic integrators [6-8] be properly used to achieve symmetric seperation of integrable Lie transformations so that spurious errors can be as much suppressed as desired.

For $i \geq 5$, since $\left(: g_i^{(n_1)} : g_i^{(n_2)}\right)$ is a homogenous polynomial with degree higher than 7, a factorization with up to the 7th order is easily obtained by directly using the first-order integrator,

$$\exp\left(\sum_{n=1}^{N_i} : g_i^{(n)} :\right) = \prod_{n=1}^{N_i} \exp\left(: g_i^{(n)} :\right) + \epsilon(2i-2), \quad (19)$$

where $i \geq 5$ and $\epsilon(2i-2)$ represents the truncated terms, which are homogeneous polynomial with degree higher than $2i-3$. For $i = 5$ and 6, the lowest-order truncated term is a homogeneous polynomial of degree 8 and 10, respectively.

For homogeneous polynomials of degree 4, we use the 2nd-order integrator and obtain a 7th-order symplectic map

$$\exp\left(\sum_{n=1}^{20} : g_4^{(n)} :\right) = \left(\prod_{i=1}^{20} e^{:\frac{1}{2} g_4^{(n_i)}:}\right) \prod_{i=1}^{20} e^{:\frac{1}{2} g_4^{(21-n_i)}:} + \epsilon(8), \quad (20)$$

where $(n_1, n_2, \ldots, n_{20})$ is any permutation of $(1, 2, \ldots, 20)$. The lowest-order truncated term in Eq. (20) is a homogeneous polynomial of degree 8.

In order to obtain a 6th-order symplectic map, we have to use the 4th-order integrator [6-8] to factorize $\exp(: f_3 :)$, which yields a product of $7^3 = 343$ Lie transformations associated with integrable polynomials:

$$\exp\left(\sum_{n=1}^{8} : g_3^{(n)} :\right) = \prod_{i=1}^{7}\prod_{j=1}^{7}\prod_{k=1}^{7} \exp\left(: d_i d_j d_k D_{ijk} :\right) + \epsilon(7). \quad (21)$$

$$d_1 = d_7 = \frac{1}{2(2-2^{1/3})}, \quad d_2 = d_6 = \frac{1}{2-2^{1/3}}, \quad d_3 = d_5 = \frac{1-2^{1/3}}{2(2-2^{1/3})}, \quad d_4 = \frac{-2^{1/3}}{2-2^{1/3}}. \quad (22)$$

D_{ijk} is an integrable polynomial of degree 3 that can be chosen according to following pattern,

$$i = \text{even} \begin{cases} j = \text{even} \begin{cases} k = \text{even}, & D_{ijk} = g_3^{(n_1)} \\ k = \text{odd}, & D_{ijk} = g_3^{(n_2)} \end{cases} \\ j = \text{odd} \begin{cases} k = \text{even}, & D_{ijk} = g_3^{(n_3)} \\ k = \text{odd}, & D_{ijk} = g_3^{(n_4)} \end{cases} \end{cases}$$

$$i = \text{odd} \begin{cases} j = \text{even} \begin{cases} k = \text{even}, & D_{ijk} = g_3^{(n_5)} \\ k = \text{odd}, & D_{ijk} = g_3^{(n_6)} \end{cases} \\ j = \text{odd} \begin{cases} k = \text{even}, & D_{ijk} = g_3^{(n_7)} \\ k = \text{odd}, & D_{ijk} = g_3^{(n_8)} \end{cases} \end{cases}$$

where $(n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8)$ is any permutation of the first eight digits, (1, 2, 3, 4, 5, 6, 7, 8). The lowest-order truncated term in Eq. (21) consists of homogeneous polynomials of degree 7.

IV. CONCLUSION

We have shown that any polynomial can be written as a sum of integrable polynomials of the same degree. The number of optimized integrable polynomials is much smaller than the number of monomials. For homogeneous polynomials of degree 3 to 6, we were able to group 56, 126, 252, and 462 monomials into 8, 20, 42, and 79 integrable polynomials, respectively. All Lie transformations associated with these integrable polynomials were translated into simple iterations that can be directly used in tracking. By utilizing the symmetric symplectic-integrators, we have developed a factorization scheme based on the integrable polynomials in which Lie transformations associated with homogeneous polynomials are converted into a product of Lie transformations associated with integrable polynomials. A much smaller number of integrable polynomials not only serves a more accurate set of factorization bases but also enables us to use high-order factorization schemes so that the truncation error can be greatly suppressed. The map in the form of Lie transformations associated with integrable polynomials could therefore be a reliable model for studying the long-term behavior of symplectic systems in the phase space region of interest.

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