An Optimized Formulation for Deprit-Type Lie Transformations of Taylor Maps for Symplectic Systems

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Abstract

We present an optimized iterative formulation for directly transforming a Taylor map of a symplectic system into a Deprit-type Lie transformation, which is a composition of a linear transfer matrix and a single Lie transformation, to an arbitrary order.

For a sympletic system, a one-turn map can be written as a composition of a linear transfer matrix and a nonlinear Taylor map M of the form [1]

$$M\vec{z} = \vec{z} + \vec{U}_2(\vec{z}) + \vec{U}_3(\vec{z}) + \ldots \tag{1}$$

which can be converted order-by-order into Lie transformations in the form of Dragt-Finn factorization [2]:

$$M\vec{z} = e^{:f_3(\vec{z}):}e^{:f_4(\vec{z}):}\ldots\vec{z}\,, \tag{2}$$

where $\vec{z}$ represents the canonical phase-space coordinates; $f_i(\vec{z})$ and $\vec{U}_i$ are the homogeneous polynomial and the vectorial homogeneous polynomial of degree i, respectively; $:f_i(\vec{z}):$ is the Lie operator associated with the function $f_i(\vec{z})$, which is defined by the Poisson bracket operation $:f_i(\vec{z}):\vec{z} = [f_i(\vec{z}), \vec{z}]$. By means of the Campbell-Baker-Hausdorff (CBH) formula [2], the product of Lie transformations in Eq. (2) can be combined to form a single Lie transformation:

$$M\vec{z} = e^{:g(\vec{z}):}\vec{z}\,, \tag{3}$$

where

$$g(\vec{z}) = g_3(\vec{z}) + g_4(\vec{z}) + \ldots\,, \tag{4}$$

and $g_i(\vec{z})$ is a homogeneous polynomial of order i. Note that except $g_3(\vec{z}) = f_3(\vec{z})$, $g_i(\vec{z})$ is generally different from $f_i(\vec{z})$. Since obtaining a single Lie transformation from Eq. (2) via CBH formula is pretty tedious and one may need such a single Lie transformation under certain circumstances [3], we have worked out an optimized algorithmic formulation for obtaining this single Lie transformation directly from the Taylor map of Eq. (1) [4]. It should be noted that we are not claiming that we are the first to try such a direct single Lie transformation. It is very likely that others may have different approach. The purpose of this note is to share with colleagues the simple and optimized algorithm we have obtained. The algorithm is described as follows.

Let us define, for each order n, a set of auxiliary vector homogeneous polynomials of degree n, $\{\vec{W}_n^{(m)}(\vec{z}), m = 1, 2, \ldots, n\}$. $g_{n+1}(\vec{z})$ for $n = 2, 3, \ldots$ are then obtained through order-by-order iteration given by the following steps:

$$g_{n+1}(\vec{z}) = -\frac{1}{n+1}\vec{z}^{\,T} S\vec{W}_n^{(1)}(\vec{z})\,, \tag{5}$$

where

$$\vec{W}_2^{(1)}(\vec{z}) = \vec{U}_2(\vec{z}), \tag{6}$$

and for $n \geq 3$,

$$\vec{W}_n^{(1)}(\vec{z}) = \vec{U}_n(\vec{z}) - \sum_{m=2}^{n-1}\vec{W}_n^{(m)}(\vec{z}), \tag{7}$$

where $\vec{W}_n^{(m)}$ for $2 \leq m \leq n$ is given by

$$\vec{W}_n^{(m)}(\vec{z}) = \frac{1}{m}\sum_{i=1}^{n-m} :g_{i+2}(\vec{z}): W_{n-i}^{(m-1)}(\vec{z}). \tag{8}$$

In Eq. (5), S is the antisymmetric matrix [1] and the superscript T denotes the transpose.

This optimized algorithm is planned to be implemented in Zlib [5], a differential Lie algebraic numerical library.

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