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SYNCHROBETATRON COUPLING EFFECTS IN ALTERNATING PHASE FOCUSING*

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Abstract

In the alternating phase focusing linac with a symmetric synchronous phase sequence, the lowest order resonance ($\sigma_\ell - 2\sigma_t = 0$) due to the synchrobetatron coupling occurs naturally, causing significant emittance transfer between the longitudinal and transverse motions. A model for the coupled motion including the amplitude dependent tune shift is proposed. Two approximate invariants are derived. Results from computer simulations of a bunched beam yield good agreement with the formula for the invariant derived from the theory. We provide, moreover, a way to move the parameters away from the lowest order resonance, and thus lower the emittance exchange due to the resonance.

I. Introduction

It is well known that longitudinal stability can be obtained in a non-relativistic drift tube accelerator by traversing each gap as the rf accelerating field rises. However, the rising accelerating field leads to a transverse defocusing force which is usually overcome by the use of magnetic focusing elements inside the drift tubes. Alternating the sign of the synchronous phase is a way to provide both longitudinal and transverse focusing without the use of focusing magnets. Exploration of this idea [1] shows that the stable longitudinal phase space area which is related to the current carrying capacity is smaller than for the Alvarez type DTL. In an earlier paper [2], we tested the current carrying capacity of an APF linac by adapting the simulation code PARMILA, which includes space charge, to the APF structure. We found, however, that significant emittance growth arose even in a low intensity beam for which phase space matching was approximately achieved. In the present paper, we show that the emittance growth for a matched beam without space charge is a response to the lowest order resonance ($\sigma_\ell = 2\sigma_t$) which naturally occurs in symmetric APF [3].

In the following study, we assume a synchronous phase configuration of the periodic length $N\beta\lambda$ where the synchronous phase pattern is

$$\phi_{s(i)} = -\phi_0 - \phi_1,\ i = 1, 2, \cdots, N/2,$$
$$\phi_{s(i)} = -\phi_0 + \phi_1,\ i = N/2+1, \cdots, N.$$

Here, both ϕ_0 and ϕ_1 are positive, with ϕ_0 representing a small asymmetric offset to accompany the large alternating ϕ_1. The desired synchronous phase configuration can be obtained by choosing drift tube lengths which alternate appropriately. In previous work [2], where no synchrobetatron coupling was considered, we concluded that, to achieve simultaneous beam matching in both directions while still keeping the phase acceptance and transverse beam size constant, we must separately keep ϕ_0, $K\cos\phi_1$, $KN\sin\phi_1$ continuous across any transition, where K is a dimensionless parameter defined by $K \equiv 2\pi ZeE_0T\lambda/Amc^2\beta\gamma^3$. We then have to increase the average accelerating field E_0T as $\beta\gamma^3$ to keep K as well as ϕ_0, ϕ_1 and N constant. Another possibility is to modulate the synchronous phase ϕ_1 continuously such that the effective N increases continuously as $\beta\gamma^3$, keeping ϕ_0 and E_0T constant. Here we study the first scheme only.

II. Analysis and Simulation

We assume that the change of $\beta\gamma$ is small over a focusing period and adopt the E_0T-ramping scheme which makes K constant. For an APF structure, the equations for the transverse and longitudinal motions can be derived by a Fourier expansion of the step functions [1]. Let us consider the smoothed version of the equations of motion containing the coupling terms $O(x^2)$ and $O(x\psi)$, and the terms $O(\psi^2)$ and $O(\psi^3)$ which yield the longitudinal amplitude dependent tune shift:

$$x'' + \sigma_t^2 x = -\sigma_a^2 x\psi, \tag{1.1}$$

$$\psi'' + \sigma_\ell^2\psi = \sigma_b^2\psi^2 + \sigma_d^2\psi^3 - \frac{k_w^2}{2}\sigma_c^2 x^2, \tag{1.2}$$

where $\psi = \phi - \phi_s$, $k_w = 2\pi/\beta\gamma\lambda$, $\tau = s/N\lambda\beta$, $' = d/d\tau$. Here

$$\sigma_t^2 = B_s + \frac{1}{8\pi^2}\sum_{n=1}^{\infty}\left[\frac{C_{s(n)}}{n}\right]^2, \tag{2.1}$$

$$\sigma_\ell^2 = -2B_s + \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\left[\frac{C_{s(n)}}{n}\right]^2, \tag{2.2}$$

$$\sigma_a^2 = B_c\left\{1 - \frac{1}{16\pi^4}\sum_{n=1}^{\infty}\left[\frac{C_{s(n)}}{n^2}\right]^2\right\} - \frac{1}{8\pi^2}\sum_{n=1}^{\infty}\frac{C_{s(n)}C_{c(n)}}{n^2}, \tag{2.3}$$

$$\sigma_b^2 = B_c\left\{1 + \frac{1}{8\pi^4}\sum_{n=1}^{\infty}\left[\frac{C_{s(n)}}{n^2}\right]^2\right\} - \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{C_{s(n)}C_{c(n)}}{n^2}, \tag{2.4}$$

$$\sigma_c^2 = B_c\left\{1 + \frac{1}{32\pi^4}\sum_{n=1}^{\infty}\left[\frac{C_{s(n)}}{n^2}\right]^2\right\} + \frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{C_{s(n)}C_{c(n)}}{n^2}, \tag{2.5}$$

$$\sigma_d^2 = -\frac{B_s}{3} + \frac{1}{4\pi^2}\sum_{n=1}^{\infty}\left[\frac{C_{s(n)}}{n}\right]^2 - \frac{1}{8\pi^4}\sum_{n=1}^{\infty}\left[\frac{C_{s(n)}}{n^2}\right]^2 - \frac{1}{64\pi^6}\sum_{n=1}^{\infty}\left[\frac{C_{s(n)}^2}{n^3}\right]^2. \tag{2.6}$$

The dimensionless parameters in Eqs. (2) are

$$B_s = \frac{KN}{2}\sum_{i=1}^{N}\sin\phi_{s(i)},$$

$$B_c = \frac{KN}{2}\sum_{i=1}^{N}\cos\phi_{s(i)},$$

$$C_{s(n)} = KN\sqrt{\left[\sum_{i=1}^{N}\cos 2n\pi\tau_i \sin\phi_{s(i)}\right]^2 + \left[\sum_{i=1}^{N}\sin 2n\pi\tau_i \sin\phi_{s(i)}\right]^2},$$

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$$C_{\alpha(n)} = KN\sqrt{\left[\sum_{i=1}^{N}\cos 2n\pi\tau_i \cos\phi_{s(i)}\right]^2 + \left[\sum_{i=1}^{N}\sin 2n\pi\tau_i \cos\phi_{s(i)}\right]^2},$$

where the τ_i coordinate is

$$\tau_1 = 0, \quad \tau_i(i \neq 1) = \sum_{j=1}^{i-1}(2\pi + \phi_{s(j+1)} - \phi_{s(j)})/2\pi N.$$

The nonlinear coupling between x and ψ motions can easily be seen here. The second order solutions for Eqs. (1) include the resonance terms, containing the factor $\Delta_- = 2\sigma_t - \sigma_\ell$ in the denominator. When $\phi_0 = 0$, we have $B_s = 0$ and thus $\Delta_- =0$, because of Eqs. (2.1) and (2.2). Thus the resonance due to the synchrobetatron coupling, which will cause rapid emittance exchange, is expected to occur when a symmetric phase sequence is chosen. Other higher order resonances can also exist when taking the higher order coupling terms into account. However, the (2,1) difference resonance described above is the lowest order and the most severe one, as long as there is no troublesome sum resonance.

Upon treating the coupling and nonlinear oscillation terms on the right hand sides of Eqs. (1) as the forcing terms that drive the linear oscillations close to resonance, we may try the solutions of Eqs. (1) similar to the solutions of the linear system, but with slowly varying amplitudes and phases. Specifically, we write

$$x(\tau) = X(\tau)\cos(\sigma_t\tau + \alpha(\tau)), \tag{3.1}$$

$$\psi(\tau) = \Phi(\tau)\cos(\sigma_\ell\tau + \beta(\tau)), \tag{3.2}$$

with the conditions:

$$x'(\tau) = -\sigma_t X(\tau)\sin(\sigma_t\tau + \alpha(\tau)), \tag{4.1}$$

$$\psi'(\tau) = -\sigma_\ell\Phi(\tau)\sin(\sigma_\ell\tau + \beta(\tau)). \tag{4.2}$$

The first order derivatives of $X(\tau)$, $\Phi(\tau)$, $\alpha(\tau)$ and $\beta(\tau)$ can be obtained by substituting Eqs. (3) into Eqs. (1) under the assumptions of Eqs. (4), and can be approximated by taking their averages over one cycle of rapid variation, according to the KBM (Krylov-Bogoliubov-Mitropolsky) method. The high frequency modes are averaged out and only the low frequency Δ_- part near resonance is kept. Implementation of the KBM averaging transforms the two second order equations into four first order equations, and two integrals of motion can then be derived. We relate the averaged maximum amplitudes $X(\tau)$ and $\Phi(\tau)$ to the normalized emittances for particle beams with uniform distributions in both the longitudinal and transverse directions: $X^2(\tau) = \varepsilon_x N\lambda/\sigma_t$ and $\Phi^2(\tau) = \varepsilon_z k_w^2 N\lambda/\sigma_\ell$, where ε_x is the normalized transverse effective emittance in x-p_x phase space, and ε_z is the normalized longitudinal effective emittance in z-p_z phase space. Here we define the effective emittance as four times of the rms emittance for the upright ellipse and treat $\gamma \approx 1$ as the non-relativistic approximation. The two adiabatic invariants E_1 and E_2 thereby are

$$E_1 = \sigma_t X^2(\tau)/\sigma_a^2 + 2\sigma_\ell\Phi^2(\tau)/k_w^2\sigma_c^2$$
$$= N\lambda(\varepsilon_x/\sigma_a^2 + 2\varepsilon_z/\sigma_c^2), \tag{5.1}$$

$$E_2 = \Delta_- J + \eta_2 J^2/2 - \eta_1(1-J)\sqrt{J}\cos\Psi, \tag{5.2}$$

where

$$J(\tau) = 2\sigma_\ell\Phi^2(\tau)/\sigma_c^2 k_w^2 E_1 = 2N\lambda\varepsilon_z/\sigma_c^2 E_1, \tag{6.1}$$

$$\Psi(\tau) = \Delta_-\tau + 2\alpha(\tau) + \beta(\tau), \tag{6.2}$$

$$\eta_1 = (k_w\sigma_a^2/2\sigma_t)\sqrt{\sigma_c^2 E_1/2\sigma_\ell},$$

$$\eta_2 = \left(\frac{\sigma_c^2 k_w^2 E_1}{2\sigma_\ell}\right)\left(\frac{3\sigma_d^2}{8\sigma_\ell} + \frac{5\sigma_b^4}{12\sigma_\ell^3}\right),$$

where the term in σ_b^4 has been carried to 2nd order for consistency in obtaining the J dependence of the tune shift. Note that the second invariant E_2 can be determined from the initial conditions of the variables J and Ψ at the injection point of a linac, and is derived with the assumption of constant β to treat the parameters η_1 and η_2 as constant. The simulation results in Fig. 1 show the transfer of normalized rms emittance between the longitudinal and transverse motions and also confirm the validity of Eq. (5.1). The saturation of emittance exchange, i.e., de-coupling of the coupled oscillation, is due to the effect of acceleration which reduces the coupling strength. A plot of $\cos\Psi$ as a function of J gives a qualitative description of the behavior of the system. Fig. 2 shows an example of the configuration diagram for Eq.(5.2). The stable zone of the diagram shown is within the region $-1 \leq \cos\Psi \leq 1$ and $0 \leq J \leq 1$. The coupled oscillations are bounded due to the fact that the first invariant is the sum of square of amplitudes, which is, per se, the property of a difference resonance. In Fig. 2, all initial coupling phase angles $\Psi(0)$ are employed equally from 0 to 2π when considering the particle beam with a uniform distribution of amplitudes X and Ψ in the trace space. The rise or fall of an emittance oscillation at the beginning depends upon the initial amplitude J(0).

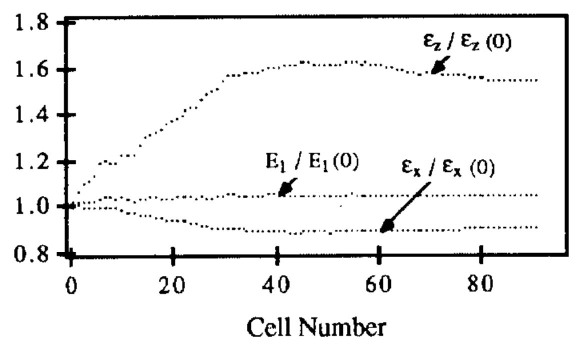

Fig. 1 Simulation results for the emittance transfer between longitudinal and transverse motion, and the first invariant from Eq. (5.1). The parameters are K=0.556, N=4, ϕ_1 = 70, ϕ_0 = 0. The injection emittances are: ε_x = 0.5242 cm-mrad, ε_z = 0.0758 cm-mrad.

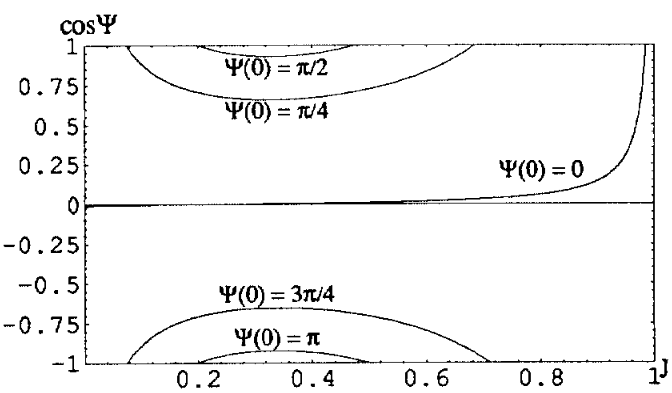

Fig. 2 Configuration diagram for $\cos\Psi$ vs. J.

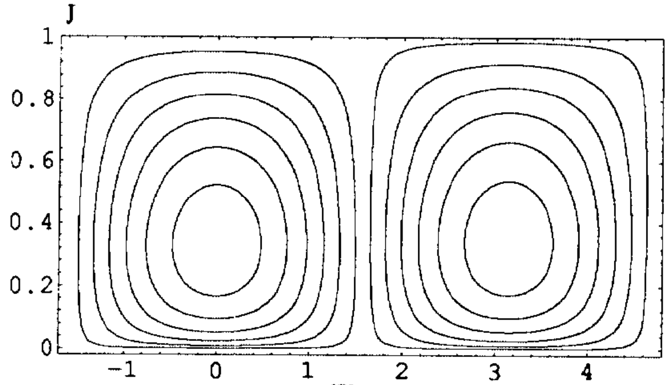

Fig. 3 Poincare mapping of J-Ψ space, when $\phi_0 = 0$.

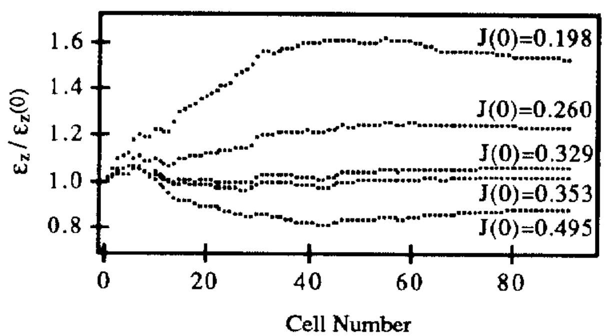

Fig. 4 Simulation results for the longitudinal emittance growth with different initial scaled emittance J(0), where the system is on resonance.

The existence of the first invariant enables us to easily reduce the four differential equations to two with the help of the function J:

$$J'(\tau) = -\eta_1(1-J(\tau))\sqrt{J(\tau)}\sin\Psi(\tau), \tag{7.1}$$

$$\Psi'(\tau) = \Delta_- + \eta_2 J(\tau) + \eta_1\left(\frac{3J(\tau)-1}{2\sqrt{J(\tau)}}\right)\cos\Psi(\tau). \tag{7.2}$$

It is worthwhile to note that the result of the second invariant can also be obtained by the Hamiltonian equations of motion, with $J(\tau)$ and $\Psi(\tau)$ treated as a pair of conjugate canonical variables. The Hamiltonian $H(\Psi,J)$ is then just identical to the second invariant E_2.

The phase space of the Hamiltonian flow for system (5.2) at resonance, i.e. when $\Delta_- = 0$, is plotted in Fig. 3. The two fixed points on the resonance manifold can be found via the conditions: $J'=0$ and $\Psi'=0$. Numerically, the term involving the amplitude dependent tune-shift η_2 contributes only as a small modification of the value of J at the fixed point. The fixed points of the system at resonance are then approximately equal to $J \approx 1/3$, $\Psi = 0$ and π. According to Eqs. (5) and (6), when J = 1/3, the ratio of longitudinal and transverse emittances have a following relationship:

$$\varepsilon_z/\varepsilon_x = (\sigma_c/\sigma_x)^2/4 \approx 1/4.$$

Therefore, if we inject a beam with the emittance ratio $\varepsilon_z/\varepsilon_x$ near 1/4, such that the initial J is near 1/3, which is close to the fixed point of the resonance Hamiltonian when $\phi_0 = 0$, the variation of J on average will then increase only a small amount, due to the asymmetric shape of phase space [cf. Fig. 3], until it reaches equilibrium [cf. Fig. 4]. For the particles with initial J smaller (larger) than 1/3, the average variation of J for particles with different phases will then tend to grow (decrease). Simulations of the bunched beam for different scaled emittance ratio shown in Fig. 4 confirm this prediction. The domain of resonance can be found from the stationary solution of Eqs. (7.2) when $\Psi' = 0$, Ψ=0 or π and the η_2 term is neglected: $|\Delta_-(\text{band})| \le |\eta_1(3J-1)/2\sqrt{J}|$. Therefore, when the initial J is not near 1/3, the design of an APF linac requires as large a non-zero phase offset as possible such that the detuning Δ_- is away from the resonance band, reducing the emittance exchange to as low a level as possible. Results of simulations for $J(\tau)/J(0)$ with different ϕ_0 are shown in Fig. 5.

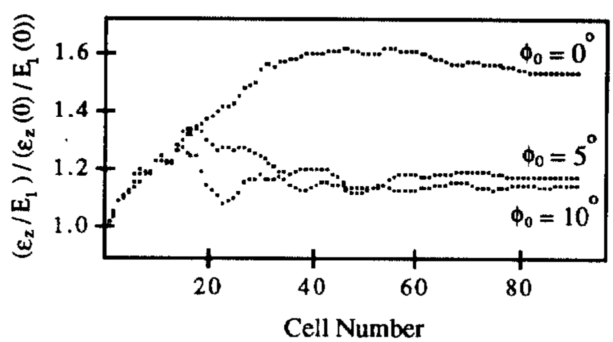

Fig. 5 Simulation results for the longitudinal emittance growth with different phase offsets.

We note that the constant-β ansatz is not physically realistic, since the acceleration rate is usually higher than the emittance exchange rate. When considering the acceleration, where the parameters η_1 and η_2 are τ dependent, the second invariant E_2 in Eq. (5.2) can no longer be achieved by integration of Eqs. (7.1) and (7.2). However, Figs. 2 and 3 can still give a qualitative description of the dynamics of synchrobetatron coupling.

III. Summary

The equations of coupled motion for an APF linac are truncated, smoothed and averaged. We have shown that in both the simulations and the analysis, the emittance exchange between the longitudinal and transverse motions due to the coupling resonance can be decreased by either choosing an emittances ratio such that the system is close to the fixed points of the resonance manifold; or by introducing a non-zero phase offset in the synchronous phase sequence of the APF linac.

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