Longitudinal Dynamics for Electrons in the Thermal Wave Model for Charged Particle Beams

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Abstract An extension of the longitudinal thermal wave model, including both radiation damping and quantum excitation (stochastic effect) is presented here. We show that, in the presence of the RF potential well, the longitudinal dynamics is governed by a 1-D Schrödinger-like equation for a complex wave function whose squared modulus gives the longitudinal bunch profile. Remarkably, the appropriate emittance scaling is naturally recovered, and the asymptotic equilibrium condition for the bunch length is found. These results open the possibility to apply the thermal wave model, already tested for protons, in a more accurate way to electrons.

1 Introduction

In the study of charged particle beam dynamics for accelerators and plasma physics, a number of nonlinear and effects are relevant [1]. Due to the electromagnetic interactions between the particles and their image charges induced in the surroundings, these nonlinear effects also acquire collective nature [1]. This property is enhanced for very intense beams, which are employed in very high luminosity colliders. In addition, radiation damping and quantum electromagnetic fluctuations (quantum excitations) are generally present in the beam longitudinal dynamics especially for electron bunches.

Recently, a *thermal wave model for charged particle beam dynamics* has been formulated [2] and successfully applied to a number of linear and nonlinear problems in beam physics [3]-[7]. In this approach, the beam transverse (longitudinal) dynamics is formulated in terms of a complex function, the so called beam wave function (bwf), whose squared modulus is proportional to the bunch density. This wave function satisfies a Schrödinger-like equation in which Planck's constant is substituted with the transverse (longitudinal) bunch emittance [2],[7]. In particular, this model is capable of reproducing the main results of the conventional theory about transverse beam optics and dynamics (in linear and nonlinear devices) [2]. Moreover, it represents a new approach to estimate the luminosity in particle accelerators [4], [5], as well as to study the self-consistent beam-plasma interaction [3]. Remarkably, this model, if applied to the longitudinal bunch dynamics, allows us to describe, in a simple way, the synchrotron motion when both self-interaction and the radio frequency (RF) potential well are taken into account. In particular, the right conditions for the coherent instability in circular machines have been recovered [6],[7] and new interesting soliton-like solutions for the beam density have been discovered [6],[7].

In this paper we improve the thermal wave model for longitudinal bunch dynamics given in [6], [7]. By starting from the conventional longitudinal single-particle dynamics in circular accelerators, the problem that we want to solve is formulated in terms of an appropriate wave model which describes the evolution of the beam, when the RF potential well is taken into account together with radiation damping and quantum excitation. We show that the longitudinal beam dynamics is still correctly governed by a Schrödinger-like equation for the bwf. The envelope description is straightforwardly obtained from the wave solution and, correspondingly, the results are compared with those that are given in the conventional theory. In particular, an asymptotic time-limit for the bunch length and the emittance time-scaling law are obtained.

2 Definition of the problem

It is well known that the motion of a single particle within a stationary bunch travelling in a circular accelerating machine with velocity βc ($\beta \approx 1$), with radius $R_0 = cT_0/2\pi$ (T_0 being the revolution period), if both radiation damping and quantum excitation are taken into account, is governed by the following coupling equations:

$$\frac{dx}{ds} = \eta \mathcal{P} \quad , \tag{1}$$

$$\frac{d\mathcal{P}}{ds} = -\frac{q\Delta V}{cT_0 E_0} - \frac{D}{cT_0}\mathcal{P} - \frac{1}{E_0}\frac{dR}{ds} \quad , \tag{2}$$

where $\mathcal{P} \equiv \frac{\Delta E}{E_0}$ is the relative longitudinal energy spread of the particle with respect to the synchronous particle ($\Delta E = 0$), and $U(x,s) \equiv (1/cT_0E_0)\int_0^x q\Delta V\, dx'$ is the effective potential energy that the particle sees after a turn in the ring (ΔV being the corresponding total voltage variation seen by the particle). x is the longitudinal displacement of the particle in the bunch with respect to the synchronous one and $s \equiv ct$ (t being the time). E_0 and q

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stands for the synchronous particle energy and charge, respectively. $\eta \equiv \alpha - 1/\gamma^2$ are the phase slip factor, where α and $\gamma = (1-\beta^2)^{-1/2}$ the momentum compaction and the relativistic gamma factor, respectively. D is the damping coefficient [8] and dR/ds accounts for the quantum excitation effect (noise) (R being the difference between the energy effectively radiated by the particle during a time interval t and the average of this energy). It is easy to see that (1) and (2) are the usual equations for the longitudinal motion [8] under the substitution $s = ct$ and $\mathcal{P} = \Delta E/E_0$. By denoting with $< u^2 >^{1/2}$ the quantum fluctuations associated to the noise, the term dR/ds can be written in the following way:

$$\frac{dR}{ds} = \frac{\Gamma}{c} < u^2 >^{1/2} \quad , \tag{3}$$

where Γ is the mean rate of photon emission.

By considering a linearized RF-voltage only ($U_{RF} \equiv \frac{1}{cT_0E_0}\int_0^s \Delta V ds' \approx \frac{1}{2}\frac{K}{\eta}x^2$, where K is the RF cavity strength), it is easy to prove that the Lagrangian associated to (1) and (2) is given by

$$\mathcal{L}(x, x', s) = \left[\frac{1}{2\eta}x'^2 - \frac{1}{2}\frac{K}{\eta}x^2 - \left(\frac{1}{E_0}\frac{dR}{ds}\right)x\right]e^{\gamma s}, \tag{4}$$

where $x' \equiv dx/ds$ and $D/(cT_0)$ is the damping rate. Consequently, the corresponding Hamiltonian is defined as $H(x,p,s) = x'p - \mathcal{L}(x,x',s)$, with $p \equiv \frac{\partial \mathcal{L}}{\partial x'} = \frac{x'}{\eta}e^{\gamma s}$, and it can be put in the following form:

$$H(\tilde{y}, \tilde{p}, s) = \frac{\eta}{2}\tilde{p}^2 + \frac{1}{2}\frac{K}{\eta}\tilde{y}^2 - \frac{1}{2}\frac{K}{\eta}\tilde{x}_0^2(s) \quad . \tag{5}$$

where $\tilde{x}_0(s) \equiv -\frac{\eta e^{\frac{\gamma}{2}s}}{KE_0}\frac{dR}{ds} \equiv x_0 e^{\frac{\gamma}{2}s}$, $\tilde{p} \equiv p\, e^{-\frac{\gamma}{2}s}$, $\tilde{x} \equiv x\, e^{\frac{\gamma}{2}s}$, and $\tilde{y} = \tilde{x} - \tilde{x}_0$. It is interesting to observe that H, in the variable $\tilde{y}$ and $\tilde{p}$, looks like the Hamiltonian of an undamped harmonic oscillator. In order to write a Schrödinger-like equation for the bwf, which describes the longitudinal dynamics of a short bunch ($\sigma << R_0$) in the presence of both radiation damping and quantum excitation, we have to write the following quantization rules, in complete analogy with our previous works [2]-[7] $\tilde{p} \rightarrow \hat{\tilde{p}} \equiv -i\epsilon\frac{\partial}{\partial \tilde{x}}$, and $H \rightarrow \hat{H} \equiv i\epsilon\frac{\partial}{\partial s}$. Consequently, (5) gives (for $\eta \neq 0$)

$$i\epsilon\eta\frac{\partial \tilde{\Psi}}{\partial s} = -\frac{\epsilon^2\eta^2}{2}\frac{\partial^2\tilde{\Psi}}{\partial \tilde{y}^2} + \frac{1}{2}K\tilde{y}^2\tilde{\Psi} + \frac{1}{2}K\tilde{x}_0^2\tilde{\Psi} \quad , \tag{6}$$

where $\tilde{\Psi} = \tilde{\Psi}(\tilde{y}, s)$ satisfies the normalization condition $\int_{-\infty}^{\infty}|\tilde{\Psi}(\tilde{y},s)|^2\, d\tilde{y} = 1$.

3 Solutions

In order to solve (6), we introduce the following transformation for the bwf: $\tilde{\Psi}(\tilde{y},s) = \Psi(\tilde{y})\exp\left[i\frac{K}{\epsilon\eta}\int_0^s \tilde{x}_0^2(s')ds'\right]$. Thus, we easily obtain

$$i\epsilon\frac{\partial \Psi}{\partial s} = -\frac{\epsilon^2\eta^2}{2}\frac{\partial^2\Psi}{\partial \tilde{y}^2} + \frac{1}{2}K\tilde{y}^2\Psi \quad . \tag{7}$$

Solutions of (7) are well known in terms of Hermite-Gauss modes [7]:

$$\Psi_m(\tilde{y}, s) = \frac{\exp\left[-\frac{\tilde{y}^2}{4\tilde{\sigma}_y^2(s)}\right]}{\left[2\pi 2^{2m}(m!)^2\tilde{\sigma}_y^2(s)\right]^{1/4}} H_m\left(\frac{\tilde{y}}{\sqrt{2}\tilde{\sigma}_y(s)}\right) \times \exp\left[i\frac{\tilde{y}^2}{2\epsilon\eta\tilde{\rho}_y(s)} + i(1+2m)\tilde{\phi}_y(s)\right] \quad . \tag{8}$$

where the H_m's are the Hermite polynomials ($m = 0, 1, 2, ...$), and the function $\tilde{\sigma}_y(s)$ satisfies the following differential equation:

$$\frac{d^2\tilde{\sigma}_y}{ds^2} + K\tilde{\sigma}_y - \frac{\epsilon^2\eta^2}{4\tilde{\sigma}_y^3} = 0 \quad . \tag{9}$$

with $\frac{1}{\tilde{\rho}_y} = \frac{1}{\tilde{\sigma}_y}\frac{d\tilde{\sigma}_y}{ds}$, and $\frac{d\tilde{\phi}_y}{ds} = -\frac{\epsilon\eta}{4\tilde{\sigma}_y^2}$.

Note that $\tilde{\sigma}_y(s) = \left[\int_{-\infty}^{\infty}\tilde{y}^2|\Psi(\tilde{y},s)|^2\, d\tilde{y}\right]^{1/2} \equiv < \tilde{y}^2 >^{1/2}$ (quantum-like expectation value of $\tilde{y}$). Thus, Eq.(9) is the bunch envelope equation. We observe that $\tilde{\sigma}_y^2 =< (\tilde{x}-\tilde{x}_0)^2 > =< (x-x_0)^2 > e^{\gamma s}$. Consequently, we can write both the solution for the b.w.f. in terms of x and s, and the envelope equation for the quantity $\overline{\Delta x} \equiv < (x-x_0)^2 >^{1/2}$ simply from (8) and (9). We easily get:

$$\Psi_m(x, s) = \frac{\exp\left[-\frac{(x-x_0)^2 e^{-\gamma s}}{4\overline{\Delta x}^2(s)} + i\theta(x,s)\right]}{\left[2\pi 2^{2m}(m!)^2\overline{\Delta x}^2(s)\right]^{1/4}} \times H_m\left(\frac{(x-x_0)e^{-\frac{\gamma}{2}s}}{\sqrt{2}\,\overline{\Delta x}}\right) \tag{10}$$

$$\frac{d^2\overline{\Delta x}}{ds^2} + \gamma\frac{d\,\overline{\Delta x}}{ds} + \left(K + \frac{\gamma^2}{4}\right)\overline{\Delta x} - \frac{\epsilon^2\eta^2 e^{-2\gamma s}}{4\,\overline{\Delta x}^3} = 0 \quad , \tag{11}$$

with $\frac{1}{\rho_x} = \left(1/\overline{\Delta x}\right)\frac{d\,\overline{\Delta x}}{ds}$, $\frac{d\phi_x}{ds} = -\frac{\epsilon\eta}{4\,\overline{\Delta x}^2}$, and $\theta(x,s) \equiv \frac{(x-x_0)^2}{2\epsilon\eta/\left(\frac{\gamma}{2}+\frac{1}{\rho(s)}\right)} + i(1+2m)\int_0^s \frac{d\phi_x}{ds'}e^{-\gamma s'}ds'$. In particular, from the (10) we can obtain the longitudinal bunch profile, for $m = 0$ (fundamental mode), which is a pure Gaussian distribution:

$$|\Psi_0(x,s)|^2 = \frac{\exp\left[-\frac{(x-x_0)^2 e^{-\gamma s}}{2\,\overline{\Delta x}^2(s)}\right]}{\left[2\pi\,\overline{\Delta x}^2(s)\right]^{1/2}} \quad . \tag{12}$$

We note that, when radiation damping and quantum excitation can be neglected, i.e. $\gamma = 0$ and $x_0 = 0$, respectively, (11) becomes:

$$\frac{d^2\sigma_x}{ds^2} + K\sigma_x - \frac{\epsilon^2\eta^2}{4\sigma_x^3} = 0 \quad , \tag{13}$$

with $\sigma_x \equiv < x^2 >^{1/2}$ (the bunch length), which recovers the envelope equation associated to the undamped synchrotron oscillations [7].

In general, (11) shows that: (a) a friction-like term, which is in competition with the quantum excitation effect, is introduced; (b) a synchrotron frequency shift, due to the damping effect, is introduced as well; (c) the emittance scaling law can be extrapolate, which results to be:

$$\epsilon_x(s) = \epsilon\, e^{-\gamma s} \quad . \tag{14}$$

However, we will go back to discuss on the point (c) in the next Section, where we start from the r.m.s. emittance definition and give straightforwardly the emittance scaling law. We now observe that the asymptotic limit for $\overline{\Delta x}$ can be obtained from (11). In fact, in the limit $s \to \infty$, the equilibrium solution gives $\left(\overline{\Delta x}\right)_{eq} = < (x - x_0)^2 >_{eq} = 0$. However, observing that $< (x - x_0)^2 >= \sigma_x^2(s) - x_0^2(s)$, we have $(\sigma_x)_{eq} \equiv \sigma_x(s = \infty) = \frac{|\eta|}{|K| E_0} \left(\frac{dR}{ds}\right)_{s=\infty}$. On the other hand, from (3) follows that $\frac{1}{\sqrt{K E_0}} \frac{dR}{ds} = \frac{\Gamma}{\Omega_s} \frac{<u^2>^{1/2}}{E_0} \equiv \sigma_p^N(s)$, where $\Omega_s = c\sqrt{|K|}$ is the synchrotron frequency and σ_p^N stands for the bunch energy spread due to the quantum fluctuations (quantum noise). Consequently, $(\sigma_x)_{eq} = \frac{|\eta|}{\sqrt{|K|}} \left(\sigma_p^N\right)_{eq}$, where $\left(\sigma_p^N\right)_{eq} \equiv \sigma_p^N(s = \infty)_{eq}$. By introducing the synchrotron number $\nu_s \equiv \Omega_s/\omega_0 = (c\sqrt{|K|})/\omega_0$ [1], (10) easily becomes $(\sigma_x)_{eq} = \frac{|\eta| R_0}{\nu_s} \left(\sigma_p^N\right)_{eq}$, where $\omega_0 = 2\pi/T_0 = 2\pi R_0/c$ is the revolution angular frequency.

4 Emittance scaling law

First of all, it is easy to prove that, under the transformation $\tilde{p} \equiv p\, e^{-\frac{\gamma}{2}s}$, and $\tilde{x} \equiv x\, e^{\frac{\gamma}{2}s}$, the quantity $A =< \tilde{x}^2 >< \tilde{\hat{p}}^2 > - < \tilde{x}\tilde{\hat{p}} >^2$, is conserved, i.e. $< \tilde{x}^2 >< \tilde{\hat{p}}^2 > - < \tilde{x}\tilde{\hat{p}} >^2 =< x^2 >< \hat{p}^2 > - < x\hat{p} >^2$, where $< \ldots >$ stands for the quantum-like average value. Thus, by observing that $< \tilde{\hat{p}}^2 >=< \hat{p}^2 > e^{-\gamma s}$, $< \tilde{x}^2 >=< x^2 > e^{\gamma s}$ and $< \tilde{x}\tilde{\hat{p}} >=< x\hat{p} >$, and, using (10) for $m = 0$, we easily obtain $A = \frac{\epsilon^2}{4} = constant$, which shows that the *diffraction parameter* of our model, i.e. the longitudinal emittance, is one of the Courant-Snyder invariants.

We now show the connection between this invariant and the r.m.s. emittance $\overline{\epsilon^2}_{th}$, defined in the following way, which is similar to the definition given by Lapostolle [1]:

$$\overline{\epsilon^2}_{th}(s) = 4\left[< x^2 >< x'^2 > - < x x' >^2\right] \quad . \tag{15}$$

Substituting $x' = \eta p\, e^{-\gamma s}$, we find:

$$\overline{\epsilon^2}_{th}(s) = 4\left[< x^2 >< \hat{p}^2 > - < x\hat{p} >^2\right] e^{-2\gamma s} \quad , \tag{16}$$

which is the same emittance scaling law predicted by (14):

$$\overline{\epsilon^2}_{th}(s) = \epsilon^2 e^{-2\gamma s} \quad . \tag{17}$$

Eq.(17) shows that ϵ is just the initial ($s = 0$) value of the r.m.s. emittance, namely the value produced by the bunch source at a given temperature.

5 Remarks and conclusions

In this paper we have presented an extension of the recently proposed *thermal wave model for particle dynamics* [2] to the longitudinal motion in circular accelerators when both radiation damping and quantum excitation are taken into account. We have shown that the particle dynamics in the presence of a RF potential well is governed by a 1-D Schrödinger-like equation for a complex wave function, whose squared modulus gives the longitudinal bunch profile. We have proved that the solutions for the bwf of this problem are given in terms of the well known Gauss-Hermite modes. In particular, the fundamental mode (lowest-energy mode) gives a pure Gaussian space-distribution for the particles, and the corresponding envelope equation gives an asymptotic value for the bunch length, which is expressed in terms of the quantum fluctuations (noise). In addition, the emittance scaling law has been recovered. We would like to stress that (15) is similar to the Lapostolle's definition of the r.m.s. emittance [1], but here the averages are defined in a way which is formally identical to the Quantum Mechanics. In conclusion, the above results allow us to apply the thermal wave model, already successfully applied to the undamped longitudinal dynamics (protons) [7], to the synchrotron electron motion in a more accurate way, since in this case both radiation damping and quantum excitation are not negligible.

References

[1] J.Lawson *The physics of charged-particle beams*, Clarendon Press, 2nd edition, Oxford(1988)

[2] R. Fedele and G. Miele, *A Thermal-Wave Model for Relativistic-Charged-Particle Beam Propagation*, Nuovo Cimento, **13D**, 1527 (1991)

[3] R. Fedele and P.K. Shukla, *Self-Consistent Interaction between the Plasma Wake Field and the Driving Relativistic Electron Beam*, Phys.Rev.**A44**, 4045 (1992)

[4] R. Fedele and G. Miele, *Spherical Aberrations in the Thermal Wave Model for Luminosity Estimates in Particle Accelerators*, Phys.Rev.**A46**, 6634 (1992)

[5] R. Fedele, F. Galluccio, and G. Miele *A Numerical Check of the Thermal-Wave Model for Charged Particle-Beam Dynamics*, these proceedings (1993)

[6] R. Fedele, L. Palumbo, and V.G. Vaccaro *A Novel Approach for Nonlinear Longitudinal Dynamics in Particle Accelerators*, Proc. EPAC 92 (Berlin, 24-28 March 1992) H. Henke, H. Homeyer and Ch.Petit-Jean-Genaz Ed.s, Editions Frontieres (1992), p.762

[7] R. Fedele, G. Miele, L. Palumbo, and V.G. Vaccaro *Thermal-Wave Model for Nonlinear Longitudinal Dynamics in Particle Accelerators*, Dipartimento di Scienze Fisiche, Universitá di Napoli, Rep. DSF-T-93/15 (1993), submitted to Phys. Lett. A

[8] J. Haissinski, *Exact Longitudinal Equilibrium Distribution of Stored Electrons in the Presence of Self-Fields*, Il Nuovo Cimento **18 B**, 72 (1973)