A Numerical Check of the Thermal Wave-Model for Particle-Beam Dynamics

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Abstract The recently proposed *thermal wave model for transverse particle-beam dynamics* is tested numerically in the case of propagation through a quadrupole lens with sextupole deviations. This check is performed by comparing the model predictions, obtained analytically using perturbation theory at first order, with the results of a conventional tracking code. The results of this comparison are shown here: a remarkable agreement between the prediction of the wave model and the output of the standard treatment is found, which opens up the possibility of studying transverse beam-dynamics from a novel and, hopefully, very powerful point of view.

1 Introduction

Transverse beam-dynamics in particle accelerators is generally approached by means of single-particle tracking. This allows the characteristic parameters of the machine such as tunes, chromaticities or Twiss parameters to be determined, and the stability region of phase-space, the so-called 'dynamic aperture', to be identified; this last one can only be evaluated at the cost of often very long and CPU-time consuming tracking-simulation procedures. The recently proposed *thermal wave model for relativistic charged particle beam propagation* [1] allows us to represent the beam as a whole, by means of a complex function, the so-called beam wave function (bwf), whose squared modulus is interpreted as the transverse distribution function: the response of the beam to the different linear and non-linear elements throughout the machine can then be described in terms of the evolution of the above bwf.

This model assumes that the transverse particle-beam dynamics is governed by a Schrödinger-like equation for the bwf which is analogous to the equations holding in non-relativistic quantum-mechanics and electromagnetic beam-optics, as it has been recently pointed out [1]. The thermal wave-model has already been applied successfully to the treatment of an ideal quadrupole-like lens with octupole deviations [2], as well as to the description of the nonlinear beam-plasma interaction [3].

In this paper, after a brief review of the main properties of the thermal wave model, we apply the standard perturbation theory generally used in solving Schrödinger equation to determine the momentum distribution of a purely Gaussian incoming beam at the end of a quadrupole lens with a small sextupole deviation. The calculation is performed in thin lens approximation, up to first order in the sextupole strength. The theoretical predictions are then compared with the results of a standard tracking code.

2 The Thermal Wave Model

According to this model, the transverse dynamics of a relativistic particle-beam which travels along the z-axis with velocity βc ($\beta \approx 1$), interacts with the surrounding medium through a potential $u(\vec{r}, z)$, and suffers the thermal spreading (emittance spreading), is governed by a Schrödinger-like equation for a complex wave function $\Psi(\vec{r}, z)$ called the *beam wave function* (bwf). In this equation, the role of the diffraction parameter is played by the transverse emittance ϵ and the analogous of time is represented by the longitudinal coordinate z. Without lack of generality, we can consider only one transverse dimension, say x. In this case, the beam wave equation of Ref. [1] becomes

$$i\epsilon \frac{\partial \Psi}{\partial z} = - \frac{\epsilon^2}{2} \frac{\partial^2}{\partial x^2} \Psi + U(x,z)\Psi \quad , \tag{1}$$

where $U(x,z)$ is a dimensionless potential, which, in general, should be obtained by integrating the field force equation

$$F \;=\; -m_0 \gamma \beta^2 c^2 \, \frac{\partial}{\partial x} U \quad ; \tag{2}$$

m_0 and $\gamma = (1-\beta^2)^{-1/2}$ are the particle rest mass and the relativistic gamma factor, respectively.

Denoting with $\Sigma(x,z)$ and N the transverse density and the total number of particles, respectively, the physical meaning of Ψ is given by the following relationship:

$$\Sigma(x,z) \;=\; N \, |\Psi(x,z)|^2 \tag{3}$$

where the following normalization for Ψ has been provided:

$$\int_{-\infty}^{+\infty} |\Psi|^2 \, dx \;=\; 1 \tag{4}$$

According to (3), the squared modulus of Ψ provides the transverse density profile of the beam, whilst the squared modulus of its Fourier transform, $\overline{\Psi}$, provides the corresponding momentum distribution.

The pair of coupled equations (1) and (2) describes the evolution of the particle beam and also represents a *wave description* for the charged particle beam optics in paraxial approximation.

Given the beam distribution in the configuration space, it is possible to define in complete analogy with quantum mechanics, the effective transverse beam size (r.m.s.)

$$\sigma(z) \;=\; \left[\int_{-\infty}^{+\infty} x^2 \, |\Psi|^2 \, dx \right]^{1/2} , \tag{5}$$

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and the average transverse beam momentum:

$$\sigma_p(z) = \left[\frac{\epsilon^2}{2}\int_{-\infty}^{+\infty}\left|\frac{\partial}{\partial x}\Psi\right|^2 dx\right]^{1/2} . \quad (6)$$

An *uncertainty principle*, fully similar to the uncertainty principle known in quantum mechanics, holds:

$$\sigma\,\sigma_p \geq \frac{\epsilon}{2} . \quad (7)$$

Note that the definition of ϵ commonly used in accelerator physics differs from this one by a factor 2.

In the following we solve (1) in the case of a purely Gaussian initial beam, with the transverse potential given by

$$U(x,z) = \frac{1}{2}k_1\,x^2 + \frac{1}{6}k_2\,x^3 \quad : \quad (8)$$

this corresponds to a quadrupole lens of focusing strength k_1 with sextupole aberrations of strength k_2. To this end, we start considering the simplest case of a beam passing through a pure quadrupole.

3 Beam propagation in a quadrupole

Let us consider a relativistic charged-particle beam crossing a quadrupole lens. The stationary configurations for the density profile are obtained by solving the following equation

$$i\epsilon\,\frac{\partial\Psi}{\partial z} = -\frac{\epsilon^2}{2}\frac{\partial^2}{\partial x^2}\Psi + \frac{1}{2}k_1\,x^2\,\Psi . \quad (9)$$

Fixing at $z = 0$ the dimension of the beam σ_0 and its dispersion $\alpha = -\sigma_0^2/(\epsilon\rho_0)$, we get as solutions the following discrete modes

$$\Psi_n^0(x,z) = \frac{1}{[2\pi\,2^{2n}\,(n!)^2]^{1/4}}\,H_n\left(\frac{x}{\sqrt{2}\sigma(z)}\right)$$
$$\times\exp\left[-\frac{x^2}{4\sigma^2(z)} + i\frac{x^2}{2\epsilon\rho(z)} - i(2n+1)\phi(z)\right] \quad (10)$$

where H_n are Hermite polynomials, and the functions $\sigma(z)$, $\rho(z)$ and $\phi(z)$ are defined as follows

$$\sigma(z) \equiv \sigma_0\left[\left(\cos\sqrt{k_1}z + \frac{1}{\sqrt{k_1}\rho_0}\sin\sqrt{k_1}z\right)^2 + \frac{\epsilon^2}{k_1\sigma_0^4}\sin^2\sqrt{k_1}z\right]^{1/2} , \quad (11)$$

$$\frac{1}{\rho(z)} \equiv \frac{1}{\sigma(z)}\frac{d\sigma(z)}{dz} , \quad (12)$$

$$\phi(z) \equiv \left\{\arctan\left[\left(\frac{\sigma_0^2}{\epsilon\sqrt{k_1}\rho_0^2} + \frac{\epsilon}{\sqrt{k_1}\sigma_0^2}\right)\times\tan(\sqrt{k_1}z) + \frac{\sigma_0^2}{\epsilon\rho_0}\right] - \arctan\left(\frac{\sigma_0^2}{\epsilon\rho_0}\right)\right\} \quad (13)$$

In the simple case of $|\rho_0| = \infty$ and for a thin lens $\sqrt{k_1}l$, with l the length of the lens, we get the approximated results: $\sigma(l) \approx \sigma_0$, and $\rho(l) \approx -1/(k_1 l)$.

4 Sextupole aberrations

If we consider a quadrupole lens with sextupole deviations, the equation to solve is (1) with the potential (8). Unfortunately exact solutions of this equation are not known, thus we adopt a standard perturbative technique [2]. We denote with $V(x,z) = (1/6)k_2x^3$ the sextupole potential, and treat its effect as a perturbation term with respect to the aberrationless hamiltonian contained in the r.h.s. of (9). Provided that $\sigma_0 k_2/3k_1 \ll 1$, it is easy to show that the non-normalized bwf at the exit of the lens is given by

$$\Psi(x,l) = \left(1 - i\frac{l}{6\epsilon}\,k_2x^3\right)\Psi_{in}(x,0) , \quad (14)$$

where Ψ_{in} is the initial condition for the bwf, fixed at the beginning of the lens. If we assume as initial condition a pure Gaussian beam-density profile, (10) for $n = 0$, with vanishing initial dispersion and dimension σ_0

$$\Psi_{in}(x,0) = \frac{1}{[2\pi\,\sigma_0^2]^{1/4}}\exp\left[-\frac{x^2}{4\sigma_0^2}\right] . \quad (15)$$

In order to obtain the momentum distribution of the beam (not normalized), we Fourier transform (14), and we get

$$\overline{\Psi}(p,l) = \left[1 + \frac{k_2\sigma_0^3 l}{6\epsilon}\frac{1}{\xi^3}H_3\left(\frac{p\sigma_0}{\epsilon}\frac{1}{\xi}\right)\right]\times\exp\left[-\frac{p^2\sigma_0^2}{\epsilon^2}\frac{1}{\xi^2} + i\phi(l)\right] , \quad (16)$$

$$\text{with } H_3(x) = 8x^3 - 12x$$
$$\text{and } \xi \equiv \sqrt{\left(1 + i\frac{2K_1\sigma_0^2}{\epsilon}\right)} \equiv \sqrt{1+i\delta} . \quad (17)$$

where $K_1 \equiv k_1 l$ is the integrated focusing strength. We now denote with $\tau = K_2\sigma_0^3/(6\epsilon)$ ($K_2 \equiv k_2 l$), and with $y = p\sigma_0/\epsilon \equiv x'/2\sigma_{p_0}$; the squared modulus of the normalized bwf can then be written in momentum space

$$|\overline{\Psi}(y)|^2 = \left[1 + \frac{144\tau^2}{(1+\delta^2)^2}y^2 - \frac{192\tau^2}{(1+\delta^2)^3}y^4 + \frac{64\tau^2}{(1+\delta^2)^3}y^6 - \frac{24\tau(1-\delta^2)}{(1+\delta^2)^2}y + \frac{16\tau(1-3\delta^2)}{(1+\delta^2)^3}y^3\right]$$
$$\times\frac{2\sigma_0}{\epsilon}[2\pi(1+\delta^2)(1+15\tau^2)^2]^{-1/2}\exp\left[-\frac{2y^2}{(1+\delta^2)}\right] ; \quad (18)$$

note that $|\overline{\Psi}|^2$ is the product of a gaussian function times a polynomial function of order 6, where the term of order 5 is missing.

5 Numerical Check

A numerical experiment has been carried out, and the theoretical probability distribution of (18) has been compared with the one produced by a standard tracking technique.

A simple magnetic system has been considered, made of two thin multipoles, a quadrupole of integrated strength K_1, and a sextupole of integrated strength K_2, whilst

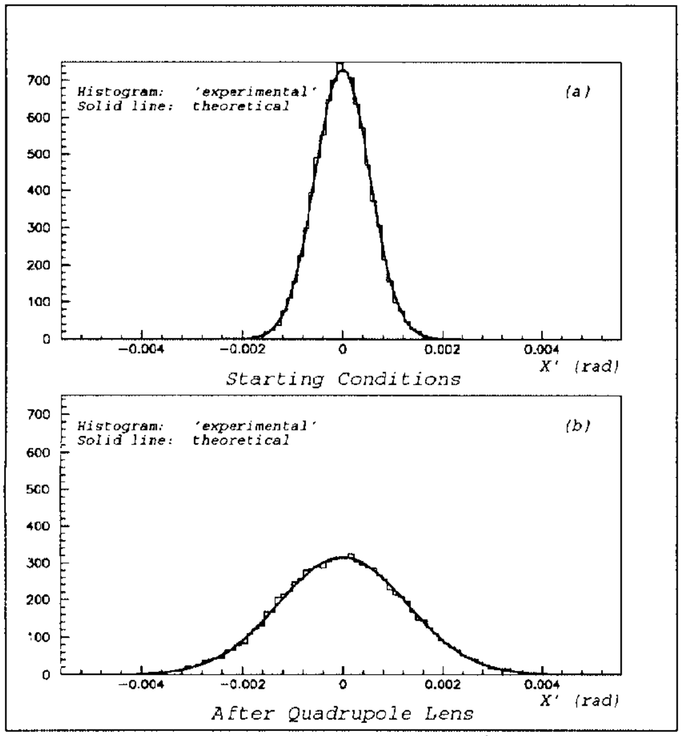

Figure 1: Simulated and Theoretical x' Distributions

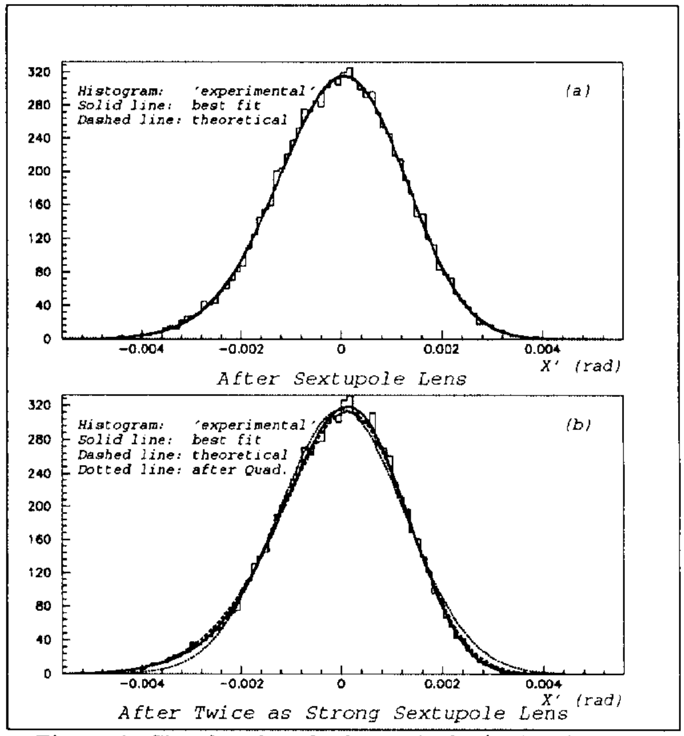

Figure 2: Simulated and Theoretical x' Distributions

30000 particles with starting coordinates x and x' randomly distributed on a 2-dimensional Gaussian have been used to simulate the beam. A tracking simulation of these particles through the device has been done by means of a simple 'kick' code: this is generally more than adequate when thin lens approximation can be applied; the coordinates of all particles have been recorded at the exit of each lens. For the sake of simplicity, it has been chosen to perform the test comparing the distributions of the x' coordinate of the phase-space ($x - x'$); comparing the x distribution would have required the addition of a drift space at the end of the apparatus.

In Fig. 1a the starting distributions of x' are displayed: the histogram, properly normalized to take into account the total number of particles and the bin width, represents the 'experimental' data, whilst the continuous curve is the theoretical starting distribution according to (18) with $K_1 = 0$ and $K_2 = 0$, i.e. $\delta = 0$ and $\tau = 0$: the agreement between the two makes us feel confident that the description of the beam by means of its σ_0 and σ_{p_0}, and the two normalizations, are done consistently.

The simulated beam distribution and the theoretical one after passing through the quadrupole, (18) with $\tau = 0$, are shown in Fig. 1b. Also here, a complete agreement is noticeable: the distributions are much wider, but they both keep purely Gaussian with same σ's and heights.

In Fig. 2 the distributions at the exit of the full device are shown for two different strengths of the sextupole: here the histogram, as usual, represents the experimental distribution, and the solid line is its best fit with a Gaussian function times a polynomial of order 6; the distribution predicted by the thermal wave model is, instead, represented by a dashed line in Fig. 2; the agreement between the predictions of the model is quite impressive: indeed only in Fig. 2b, where the sextupole strength becomes very large and the inequality $\sigma_0 k_2/3k_1 \ll 1$, is not strictly satisfied any more, the dashed line is barely visible, whilst in Fig. 2a it overlaps perfectly with the best fit curve and the only sign of its presence is the slight thickening of the line.

It should be also said that, in order to produce a detectable effect after a single pass, the sextupole strengths used exceed by far the values typical of circular accelerators.

Finally the dotted line in Fig. 2b reminds the probability distribution of the beam at the exit of the quadrupole, before undergoing the non-linear force.

6 Conclusions

This simple, but very significant numerical experiment has proved the capability of the recently proposed *thermal wave model* to describe the charged-particle beam-dynamics quite accurately.

Much remains to be done — like, for instance, the extension to 2- or even to 3-D, or the development of an iterable formulation — to make this model really interesting for the study of the typical, still unsolved, beam-dynamics problems. Nevertheless, its very innovative feature of allowing the treatment of the whole beam at the same time, makes it look extremely promising for a new, and more complete, approach to the subject.

References

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