Tune Shift Effect Due to the Sextupole Longitudinal Periodic Structure in the Superconducting Dipole Magnets

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Abstract

Using the standard Hamiltonian perturbation theory, the tune shift due to the sextupole periodic pattern in the superconducting dipole magnets is estimated for the Superconducting Super Collider (SSC) machine. The result indicates that this effect is of the order of 10^{-9}. Therefore, this effect can be neglected in the dynamics of the beam.

I. INTRODUCTION

The discovery of the sextupole, dipole, and quadrupole longitudinal periodic structure due to the persistent-current field in the HERA superconducting magnets at Deutsches-Elektronen-Synchroton Laboratory (DESY) [1] has raised questions about the possible effects of this pattern on the dynamics of the beam. Experiments carried out suggest that this periodic pattern is due to the strand pitch in the superconducting cable (s.c.), and measurements indicate that its wavelength is approximately equal to this strand pitch (9.1 ± 0.5 cm for the outer coil of the Superconducting Super Collider (SSC) s.c. dipoles). The sextupole pattern has already been confirmed in a short 50-mm R&D dipole magnet [2], and the effect in the dynamic of the beam in the SSC requires a confident estimate, even if it is known already that the effect must be small. It is possible to see this effect by calculating the tune shift through the Hamiltonian formalism. To calculate the tune shift, the superconvergent Hamiltonian perturbation method [3] is used, applying the standard canonical transformations and averaging [4].

II. HAMILTONIAN FORMALISM

The Hamiltonian for a synchronous relativistic charged particle traveling around an accelerator ring can be written as:

$$H = \frac{1}{2}(P_x - \frac{e}{cp}A_x)^2 + \frac{1}{2}(P_y - \frac{e}{cp}A_y)^2 - \frac{e}{cp}A_s + $$
$$+\frac{1}{2}K_1(s)y^2 - \frac{1}{2}\left(K_1(s) - \frac{1}{\rho^2}\right)x^2 \ , \quad (1)$$

where p is the longitudinal momentum of the particle, $P_x = p_x/p$ and $P_y = p_y/p$ are its normalized transversal momenta, e is the charge of the particle, c is the speed of light, $\rho(s)$ is the curvature of the accelerator ring, $K_1(s)$ describes the linear lattice of the machine (without longitudinal oscillation pattern), and A's are vector potential components. This Hamiltonian can be written as $H(x,y,s) = H_o(x,y,s) + V(x,y,s) + U(x,y,s)$, where H_o, $V = V^{(1)} + V^{(2)}$, and U are defined by

$$H_o(x,y,s) = \frac{1}{2}\left(P_x^2 + K_x(s)x^2\right) + \frac{1}{2}\left(P_y^2 + K_y(s)y^2\right) \ , \quad (2a)$$

$$V^{(1)}(x,y,s) = -(e/cp)\,A_s \ , \quad (2b)$$

$$V^{(2)}(x,y,s) = -(e/cp)\,(P_x A_x + P_y A_y) \ , \quad (2c)$$

and

$$U(x,y,s) = (e/cp)^2(A_x^2 + A_y^2)/2 \ . \quad (2d)$$

The longitudinal periodic structure of the magnetic field induces a longitudinal field component which, in turns, requires the three components of the vector potential. For tune shift calculations, it is more convenient to express the Hamiltonian in the canonical variable (J,ϕ), where J and ϕ are the vectors $J = (J_1,J_2)$ and $\phi = (\phi_1,\phi_2)$. This can be accomplished through the generating function

$$F(s,x,y,\phi) = -\sum_{i=1}^{2} x_i\,(\tan\phi_i - \acute{\beta}_i/2)/\beta_i(s) \ , \quad (3)$$

where $\beta_i(s)$ is the beta function associated with the motion of the particle in the i (x for $i=1$, y for $i=2$) direction; $\acute{\beta}_i$ is its derivative with respect to s, and $\phi_i(s)$ is the betatron phase, which is related to the beta function through $\phi_i(s) = \phi_i(0) + \int_0^s d\sigma/\beta_i(\sigma)$. The action, J_i, the coordinates and the canonical momenta are given by $J_i = -\partial F/\partial\phi_i = \left[x_i^2 + (\beta_i\acute{x}_i - \acute{\beta}_i x_i/2)^2\right]/2\beta_i$,

$$x_i = \sqrt{2J_i\beta_i}\cos\phi_i \ , \quad (3a)$$

$$P_i = -\sqrt{2J_i/\beta_i}\,(\sin\phi_i - \frac{1}{2}\acute{\beta}_i\cos\phi_i) \ , \quad (3b)$$

for $i = 1,2$, i.e., $i = x,y$. Furthermore, the expression (2a) becomes

$$\mathcal{H}_o = \sum_{i=1}^{2}\frac{J_i}{\beta_i(s)} \ , \quad (4)$$

and the other expressions also become functions of the action-angle variables. To calculate the tune shift, the

*Operated by the Universities Research Association, Inc., for the U.S. Department of Energy under Contract No. DE-AC35-89ER40486.

average of the Hamiltonian along the whole machine, $\mathcal{C}$, and all over the betatron phases must be determined:

$$< \mathcal{H} >= \frac{1}{2\pi}\int_0^{\mathcal{C}} ds \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} d\phi_1 d\phi_2 \, \mathcal{H}(s,\phi,J) \ . \quad (5)$$

Hence, the partial derivation of this quantity with respect to the action brings about the tune of the machine, $\nu_i = \partial < \mathcal{H} > /\partial J_i \ , i = 1,2$, which is mainly given by the average of the term (4). The other terms, (2b) to (2d), give the tune shift of this value. This expression represents the first order in perturbation strength. The sextupole component of the magnetic field does not induce tune shift of a first order in perturbation strength. To calculate its tune shift effect, it is required to go to the second order in perturbation.

To go to a second-order perturbation theory, a new canonical transformation must be made, (Φ, K). This canonical transformation is close to the identity (the original action-angle transformation (3)) and is characterised by the generating function

$$F_{new}(s,\phi,K) = \sum_{i=1}^{2} K_i\phi_i + G(s,\phi,K) \ , \quad (6)$$

where G is a function to be determined. The relation between the new variable (Φ, K) and the old ones (ϕ, J) is given by the expressions $J_i = \partial F_{neu}/\partial\phi = K_i + G_{\phi_i}$ and $\Phi_i = \partial F_{new}/\partial K_i = \phi_i + G_{K_i}$, where the subindex means partial differentiation. In addition, the new Hamiltonian, $\widehat{\mathcal{H}}(s,\Phi,K)$, is given by

$$\widehat{\mathcal{H}} = \sum_{i=1}^{2} K_i/\beta_i(s) + \sum_{i=1}^{2}(\partial G/\partial\phi_i)/\beta_i(s) + \partial G/\partial s + \\ + \mathcal{V}(s,\Phi - G_K, K + G_\phi) + U(s,\Phi - G_K, K + G_\phi) . (7)$$

Doing a Taylor expansion of the last two terms on the right hand side of (8), it follows

$$\widehat{\mathcal{H}} = \sum_{i=1}^{2} K_i/\beta_i(s) + \mathcal{V}^{(1)}_{m=2}(s,\Phi,K) + \\ + \sum_{i=1}^{2}(\partial G/\partial\phi_i)/\beta_i(s) + \partial G/\partial s + \mathcal{V} + \\ + \sum_{i=1}^{2}[\mathcal{V}_{K_i}G_{\phi_i} - \mathcal{V}_{\phi_i}G_{K_i}] + \mathcal{U}(s,\Phi,K) + \dots \ , \quad (8)$$

where a possible quadrupole term in (2) has been extracted from $\mathcal{V}$ and put together with the first-order zero average terms, first line in (8), and the term $\mathcal{U}$ has been put together with the second-order terms, third line in (8). In this expression, it is possible to make

$$\sum_{i=1}^{2}\frac{1}{\beta_i(s)}\frac{\partial G}{\partial\phi_i} + \frac{\partial G}{\partial s} + \mathcal{V} = 0 \ , \quad (9)$$

"legally" deleting the term $\mathcal{V}$ from the Hamiltonian. The soluion of this partial differential equation brings about the following expression for G (see Reference [5]):

$$G(s,\phi,K) = -\int_0^s \mathcal{V}(\xi, \phi - \psi(s) + \psi(\xi), K) \, d\xi \ , \quad (10)$$

where the components of the function ψ are defined by $\psi_i(s) = \int_0^s d\sigma/\beta_i(\sigma)$. Using this expression in (8), the full second-order approximation can be solved, neglecting higher-order terms. Consequently, the second order in perturbation Hamiltonian can be written as

$$\widehat{\mathcal{H}} = \sum_{i=1}^{2}\frac{K_i}{\beta_i(s)} + \mathcal{V}^{(1)}_{m=2}(s,\phi,K) + \mathcal{H}_{11} + \mathcal{H}_{12} + \mathcal{H}_{22} \ , \quad (11)$$

where $\mathcal{H}_{11}$, $\mathcal{H}_{12}$, and $\mathcal{H}_{22}$ are given by

$$\mathcal{H}_{11} = \sum_{i=1}^{2}\left[\mathcal{V}^{(1)}_{K_i}G^{(1)}_{\phi_i} - \mathcal{V}^{(1)}_{\phi_i}G^{(1)}_{K_i}\right] \ , \quad (12a)$$

$$\mathcal{H}_{12} = \sum_{i=1}^{2}\left[\mathcal{V}^{(1)}_{K_i}G^{(2)}_{\phi_i} - \mathcal{V}^{(1)}_{\phi_i}G^{(2)}_{K_i}\right] + \sum_{i=1}^{2}\left[\mathcal{V}^{(2)}_{K_i}G^{(1)}_{\phi_i} - \mathcal{V}^{(2)}_{\phi_i}G^{(1)}_{K_i}\right] \ , \quad (12b)$$

and

$$\mathcal{H}_{22} = \sum_{i=1}^{2}\left[\mathcal{V}^{(2)}_{K_i}G^{(2)}_{\phi_i} - \mathcal{V}^{(2)}_{\phi_i}G^{(2)}_{K_i}\right] + \mathcal{U} \ , \quad (12c)$$

where, using (2b) and (2c), the decomposition $\mathcal{V} = \mathcal{V}^{(1)} + \mathcal{V}^{(2)}$ has been made, and $G^{(i)}$ for $i = 1,2$ is defined as $G^{(i)} = -\int_0^S \mathcal{V}^{(i)}(\xi, \phi - \psi(s) + \psi(\xi), K) \ d\xi$.

III. SEXTUPOLE TUNE SHIFT

The components of the vector potential resulting from the sextupole longitudinal oscillation pattern in an s.c. dipole is given by [7]:

$$A^{(3)}_x = -(3x^2y^2 - y^4)\acute{v}/5 \ , \quad (13a)$$

$$A^{(3)}_y = +(3x^3y - xy^3)\acute{v}/5 \ , \quad (13b)$$

$$A^{(3)}_s = -(x^3 - 3xy^2)v \ , \quad (13c)$$

where $v(s)$ is the function responsible for the longitudinal oscillation pattern ($\acute{v}$ is its differentiation) and is given by $v(s) = (b_3 + a\sin\kappa s)/3$, where b_3 represents the systematic component, a represents the amplitude of the oscillation pattern, and κ denotes the wavelength number of the longitudinal periodic pattern. The contribution of the systematic sextupole component average value, b_3, is well known, and it will be ignored in the calculations.

It is not difficult to see from (13), (2), (3a), (10), and (12) the following order of dependence in the action for the second-order terms of the Hamiltonian, $\mathcal{O}(\mathcal{H}_{11}) \sim K^2$, $\mathcal{O}(\mathcal{H}_{12}) \sim K^3$, and $\mathcal{O}(\mathcal{H}_{22}) \sim K^4$. Therefore, the terms (12b) and (12c) are expected to be very small, and they will not be presented here. From the expressions (13c), (2b), (3b), and (10), the following expressions are obtained

$$\mathcal{V}^{(1)} = (e/pc)\, 2^{3/2} v \Big[(K_1\beta_1)^{3/2} \cos^3\phi_1$$

$$-3(K_1\beta_1)^{1/2} K_2\beta_2 \cos\phi_1 \cos^2\phi_2 \Big] \text{ , and} \qquad (14a)$$

$$G^{(1)} = -(e/cp)\, 2^{3/2} \Big\{ K_1^{3/2} \sum_{\rho=0}^{3} \binom{3}{\rho} \cos^{3-\rho}\phi_1 \sin^{\rho}\phi_1\, g_{11}^{\rho}(s)$$

$$-3K_1^{1/2} K_2 \sum_{\rho=0}^{1}\sum_{\acute\rho=0}^{2} \binom{2}{\acute\rho} \cos^{\sigma}\phi_1 \sin^{\rho}\phi_1 \cos^{\acute\sigma}\phi_2 \sin^{\acute\rho}\phi_2\, g_{12}^{\rho\acute\rho}(s) \Big\} , \qquad (14b)$$

where $\sigma = 1-\rho$, $\acute\sigma = 2-\acute\rho$, and the functions g_{11}^{ρ} and $g_{12}^{\rho\acute\rho}$ are given by

$$g_{11}^{\rho}(s) = \int_0^s v(\xi)\beta_1^{3/2}(\xi) \cos^{3-\rho}\delta_1 \sin^{\rho}\delta_1 \; d\xi \qquad (15a)$$

and

$$g_{12}^{\rho\acute\rho}(s) = \int_0^s v(\xi)\beta_1^{1/2}(\xi)\beta_2(\xi) \cos^{\sigma}\delta_1 \sin^{\rho}\delta_1 \cos^{\acute\sigma}\delta_2 \sin^{\acute\rho}\delta_2 d\xi \qquad (15b)$$

where δ_i is defined as $\delta_i(s,\xi) = \psi_i(s) - \psi_i(\xi)$, $i = 1, 2$. Doing the partial differentiations of (14s), calculating the obtained average values, and making some rearrangements, it follows that

$$< \mathcal{H}_{11} > = -\left(\tfrac{e}{cp}\right)^2 \Big\{ K_1^2 \left[27q_{11;1}^{1} - 9q_{11;1}^{3}\right]$$

$$+K_1K_2 \left[-9Q_{12;1}^{10} + 18Q_{12;1}^{12} - 9q_{11;2}^{1} - 9q_{11;2}^{3} + 108Q_{12;2}^{01}\right]$$

$$+K_2^2 \left[27Q_{12;2}^{10} - 9Q_{12;2}^{12}\right] \Big\} , \qquad (16)$$

where the following definitions have been used:

$$q_{11;1}^{\rho} = \frac{1}{2\pi}\int_0^C v(s)\beta_1^{3/2}(s) g_{11}^{\rho}(s)\; ds \; , \qquad (17a)$$

$$q_{11;2}^{\rho} = \frac{1}{2\pi}\int_0^C v(s)\beta_1^{1/2}(s)\beta_2(s) g_{11}^{\rho}(s)\; ds \; , \qquad (17b)$$

$$Q_{12,1}^{\rho\acute\rho} = \frac{1}{2\pi}\int_0^C v(s)\beta_1^{3/2}(s) g_{12}^{\rho\acute\rho}(s)\; ds \; , \qquad (17c)$$

and

$$Q_{12;2}^{\rho\acute\rho} = \frac{1}{2\pi}\int_0^C v(s)\beta_1^{1/2}(s)\beta_2(s) g_{12}^{\rho\acute\rho}(s)\; ds \; . \qquad (17d)$$

Thus, the tune shift is given by the partial derivation of this expression with respect to the action variables:

$$(\Delta\nu_j)_{K_i=\epsilon_N/2\gamma} = r_p\epsilon_N\lambda_j/mc^2\gamma^3 , \; j = 1, 2, \qquad (18)$$

where λ_1, λ_2 are defined as $\lambda_1 = -27q_{111}^{1} + 9q_{111}^{3} + \frac{9}{2}Q_{121}^{10} - 9Q_{121}^{12} + \frac{9}{2}q_{112}^{1} + \frac{9}{2}q_{112}^{3} - 54Q_{122}^{01}$, $\lambda_2 = 9Q_{121}^{10} - 18Q_{121}^{12} + 9q_{112}^{1} + 9q_{112}^{3} - 108Q_{122}^{01} - 27Q_{122}^{10} + 9Q_{122}^{12}$. Table 1 shows the results of these integrations along the Collider and High Energy Booster (HEB) machines of the SSC.

Table 1
Numerical Integration

25×	COLLIDER	HEB
$q_{11;1}^{1}$	-1.546×10^{15}	-5.555×10^{12}
$q_{11;2}^{1}$	-8.926×10^{14}	-1.734×10^{12}
$q_{11;1}^{3}$	-8.500×10^{14}	$+11.477 \times 10^{15}$
$q_{11;2}^{3}$	-3.363×10^{14}	$+8.616 \times 10^{15}$
$Q_{12;1}^{10}$	-9.072×10^{14}	-1.719×10^{12}
$Q_{12;1}^{12}$	-2.057×10^{11}	$+2.141 \times 10^{10}$
$Q_{12;2}^{10}$	-1.546×10^{15}	-5.479×10^{12}
$Q_{12;2}^{12}$	-7.227×10^{10}	$+1.158 \times 10^{10}$
λ_1	$+5.660 \times 10^{17}$	$+1.422 \times 10^{17}$
λ_2	$+1.895 \times 10^{17}$	$+7.805 \times 10^{16}$
$\Delta\nu_1/a^2$	2.7×10^{-10}	6.82×10^{-9}
$\Delta\nu_2/a^2$	9.1×10^{-12}	3.74×10^{-9}

As can be seen from these numerical values, the dynamics of the beam are not affected by the longitudinal sextupole oscillation patter in the s.c. magnets. It is pointed out that the values shown in the table can change by one order of magnitude, since the integration depends on the wavelength of the longitudinal oscillation pattern.

Higher-order multiples have smaller contributions than the sextupole and can be neglected as well. However, there is also a longitudinal quadrupole oscillation pattern in the s.c. dipole magnets, but since the quadrupole multiple is not a symmetry allowed in the dipole magnets, the amplitude, a, of this oscillation is expected to be random from magnet to magnet. To calculate the contribution to the tune shift of the quadrupole longitudinal oscillation pattern, a simple first order in perturbation theory can be done obtaining a contribution $\Delta\nu/a = \pm 10^{-5}$, where a must be given in Gauss/cm.

IV. CONCLUSIONS

The expected tune shift due to the longitudinal oscillation sextupole component pattern is of the order of 10^{-9}. Therefore, this pattern is not relevant for the dynamics of the particles for the SSC Collider or the HEB machines.

V. REFERENCES

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