Symplectic Scaling, a DA Based Tool

G. H. Hoffstätter and M. Berz*
Department of Physics and Astronomy and
National Superconducting Cyclotron Laboratory
Michigan State University, East Lansing, MI 48824

Abstract

Maps of magnetic particle optical elements written in geometric coordinates have two scaling properties. These are connected to the fact that the maps depend only on the ratio of field strength to magnetic rigidity and on the product of field strength to the size of the element. Once the map of an element is known for a given type of beam particles as a function of the magnetic field strength at the pole tip, the first scaling property can be used to compute the map for any particle type. With the second scaling property, the map can be computed for any similar element which differs in size. Usually the map is not known as a function of the magnetic field. With DA based programs, however, one can obtain the Taylor expansion of that function.

The expansion can serve to approximate maps which could otherwise only be calculated by very time consuming numerical integration. To make this method applicable to cases where the symplectic structure is important, canonical maps have to be approximated. The approximated maps still have to be completely symplectic up to their expansion order. To meet this requirement, we have examined how the scaling properties can be used in connection with the symplectic representations of Lie transformations and generating functions. Useful examples of the resulting symplectic scaling method include maps of fringe fields as well as solenoids. Speed and accuracy of the method, which was implemented into version 6 of COSY INFINITY, will be demonstrated and a guide given how to apply this method most efficiently.

I. Introduction

Computer codes which can manipulate and differentiate truncated power series of functions, differential algebra (DA) codes, can be used to integrate coupled autonomous differential equations $d\vec{z}/ds = \vec{f}(\vec{z})$ very efficiently by using the Lie derivative $L_f = \vec{f}\nabla + \partial_s$ [1]. This entails the possibility to obtain transfer maps of main-field regions, where the equation of motion does not depend on the independent variable s. The transfer map after a main field of length l_0 is obtained by evaluating $\vec{M}(\vec{z}) = \exp(l_0 L_f)\vec{z}$.

This method can not be applied when the equation of motion is governed by fields which depend on the path length s of the reference trajectory. Such nonautonomous differential equations are usually solved by some means of numerical integration. Evaluating this integration in DA automatically yields the transfer map [1, 2]. However, this integration is extremely time consuming compared to the method for the main field, which is faster by up to three orders of magnitude [3].

We look for an alternative which should not compromise much accuracy but work much faster. Since we want to implement the algorithm into an arbitrary order code, it should work to all orders. The obtained maps have to be completely symplectic up to their evaluation order. For repetitive systems this need is obvious. The destructive effect of symplecticity violation on phase space would be magnified with every turn [4]. The symplectic condition can also be important in single pass systems, for instance when the spherical aberration of solenoids is of interest or when an achromat is designed [6, 5], since the symplectic condition enforces certain relations between aberration coefficients.

In the past, a variety of approximations have been used which speed up the process of obtaining the desired maps, in particular for the simulation of fringe fields:

- Low accuracy numerical integration – is not accurate and not symplectic.
- Fringe field integrals, which are for instance used in the codes TRANSPORT [8] and GIOS – can not be used for solenoids, is in general not symplectic and so far only available to third order although attempts are being made to extend it to fifth order [7].
- The Impulse approximation, which is used in TRANSPORT – can not be used for solenoids and works only to second order.

Here we present an approximation without those drawbacks, which has been implemented in version 6 of the DA code COSY INFINITY.

*Supported in Part by the U.S. National Science Foundation, Grant No. PHY 89–13815, and the Alfred P. Sloan Foundation.

0-7803-1203-1/93$03.00 © 1993 IEEE

II. Symplectic Scaling

A. *Scaling*

In geometric coordinates $\vec{x} = (x, x', y, y', \delta_l, \delta_p)$, which are used in TRANSPORT [8], the transfer map has two scaling properties. Those properties are made obvious by the Lorentz force equation

$$\frac{d(\gamma m \frac{d\vec{r}}{dt})}{dt} - q\frac{d\vec{r}}{dt} \times \vec{B}(\vec{r}) = \vec{f}(\vec{r}, t) = 0 \tag{1}$$

with relativistic γ, mass m, time t, charge q, magnetic field $\vec{B}$, and coordinate vector $\vec{r}$. Let $\vec{r}(\vec{r}_0, \dot{\vec{r}}_0, t)$ be a solution of the Lorentz equation. When the field is now changed to $\alpha\vec{B}(\alpha\vec{r})$, we get a new equation of motion. This equation can be obtained by substituting $\alpha\vec{r}$ for $\vec{r}$, αt for t, and leaving $\dot{\vec{r}}$ unchanged. Therefore, a field $\alpha\vec{B}(\alpha\vec{r})$ leads to the equation

$$\vec{f}(\alpha\vec{r}, \alpha t) = 0 \quad , \tag{2}$$

which has the solution $\frac{1}{\alpha}\vec{r}(\alpha\vec{r}_0, \dot{\vec{r}}_0, \alpha t)$. This we call geometric scaling: magnifying a magnetic element and ray coordinates by a factor of α yields a possible particle ray if at the same time the field strength is reduced by the same factor.

The second scaling property, rigidity scaling, is also obtained from equation (1) and states that the particle ray does not change whenever the ratio qB/p does not change; B denotes the pole tip field.

Suppose we knew the function $\vec{x}_f = \vec{F}(\vec{x}_i, B)$, the transfer function from one plane in the accelerator to another as a function of the magnetic field at the pole tip. With geometric scaling we could find all maps for similar elements which differ in size, and with rigidity scaling all maps which differ in properties of the particle could be obtained. However, since we are interested in canonical maps, which do not scale via the above method, it is necessary to use a momentum dependent transformation $\vec{z} = T(\vec{x}, p)$ to transform from geometric coordinates $\vec{x}$ to canonical coordinates $\vec{z} = (x, a, y, b, \delta_\tau, \delta_E)$ [2].

Once a transfer map is obtained at a field B_0 in canonical coordinates by means of numerical integration in DA, it can be transformed into geometric coordinates. The transfer function in geometric coordinates contains the dependence of motion on the momentum. The required transfer function, which depends on the pole tip strength B, can therefore be created using rigidity scaling. Computing this function once for a certain particle and an element of a certain size is enough to calculate the transfer map of all kinds of particles through similar elements of any size, and hence this map contains complete information. Using DA, the Taylor expansion in the quantity $\delta_B = (B - B_0)/B_0$ around the reference field B_0 is obtained automatically. Saving this Taylor expansion gives us a reference file to approximate all kinds of maps which can be obtained by scaling. The approximation will be as accurate as the Taylor expansion approximates the function, which is very accurate for several reasons:

- The Taylor expansion in respect to δ_B can be of higher order than the order in which the map is computed.
- There are several methods which yield aberration coefficients as multiple integrals over powers of derivatives of the field and the fundamental rays [?, 6, ?]. Since the rays do not change much in fringe fields, those integrals are very close to power series in respect to B. For solenoids this is not the case since the fundamental rays in a solenoid strongly depend on B.
- The deviation of the magnetic field from the reference magnetic field is often quite benign, especially when the approach described in the next chapter is used.

This direct route yields approximate maps, which however would not be exactly symplectic. As mentioned in the introduction, this can not be tolerated. We therefore compute a symplectic representation, which depends on B, and store the Taylor expansion of this symplectic representation. Evaluating the expansion gives an approximate symplectic representation, which in turn yields a fully symplectic map.

B. *Symplectic representation*

As representation we choose the single Lie exponent, which has speed advantages compared to the other five representations that are implemented in COSY INFINITY: [9].

$$\vec{M}(\vec{z}) = M_1(B)e^{:P(B):}\vec{z} \tag{3}$$

with the usual notation $:f:g$ of the Poisson bracket of f with g, the linear matrix $M_1(B)$, and the Lie exponent $P(B)$ which is a polynomial of orders higher or equal to three in the map coordinates. The coefficients of the matrix and of the polynomial are functions of B. Therefore the map

$$M_1^{-1}(B)\vec{M}(\vec{z}, B) = \vec{z} + \vec{N}(\vec{z}, B) \tag{4}$$

has to be represented by a Lie exponent. Evaluating the symplectic condition

$$(I + \partial_{\vec{z}}\vec{N})J(I + \partial_{\vec{z}}\vec{N})^T = J \tag{5}$$

order by order shows that this representation always exists for symplectic maps and that it is unique. Here I describes the unity matrix and J the symplectic matrix.

Often there are a variety of generating functions which can represent the matrix $M_1(B)$, but it can not be guaranteed that there always exists a generating function of the classical type. For the cases of fringe fields and for solenoids, however, there is always at least one possible choice. We choose an appropriate generating function according to the greatest determinant of the submatrix which has to be inverted [1].

C. *Application*

The whole process of using the symplectic scaling (SYSCA) procedure is contained in the flow diagram in figure 1. The left part refers to the creation of a reference representation by creating a canonical reference map that contains the dependence of the map on energy and computing the dependence of the map on the magnetic field via rigidity scaling. Then the symplectic representations are computed as functions of the field strength and saved to a file. The right part refers to reading the representation and inserting δ_B suitably to describe a map $\vec{M}^s$ which can be scaled to the desired map $\vec{M}^*$. From this representation the canonical map is computed and transformed to a geometric map which is used for scaling. The scaled geometric map is finally transformed back to a canonical map.

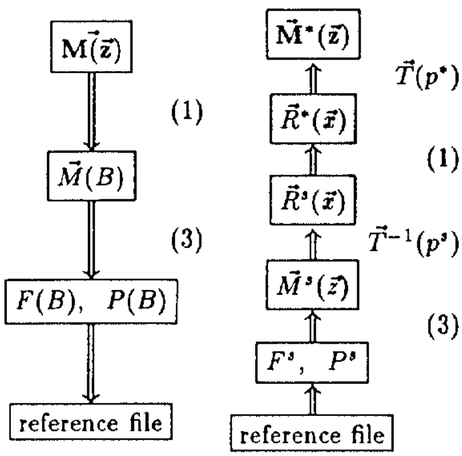

Figure 1: A map for arbitrary beam parameters, element size, and field strength can be computed from the map of a similar element using symplectic scaling.

For the reasons explained above, this approximation is very accurate for a wide range of the scaling factor δ_B. For fringe fields it can range up to 100, whereas for circular elements the tolerance is smaller for the given reasons and is usually limited to about 10%. The most efficient use of symplectic scaling does not require a great range for δ_B at all. A typical problem in the design of accelerating structures, storage rings or imaging structures is to:

- fit system parameters according to some optical conditions.
- evaluate a system with small deviations from the optimized parameters to determine the system's sensitivity to construction errors.

Both applications have in common that the map of a system has to be computed very often with slightly changed parameters. The way to tackle those problems with a gain of speed and practically no loss of accuracy is to optimize the system neglecting fringe fields. Reference files can now be created which correspond to this crude approximation as basis for the approximation of other elements. A new optimization run using SYSCA will now yield a final optimized result. The saved files can also be used as basis for the error analysis. The results will again be very accurate since the changes considered will not cause big deviations from the reference map.

Order	Solenoid	Dipole	Quadrupole
1	6	184	30
2	8	110	21
3	14	80	16
4	20	74	12
5	33	55	11
6	35	65	10
7	39	60	11

Table 1: Speed advantage of SYSCA over numerical integration.

D. *Speed and Accuracy*

Reference [3] contains several example results of SYSCA. From this and other experiences, an estimate on the speed is given in table 1 as function of the evaluation order. The accuracy depends on the order of evaluation of the expansion in δ_B and the value of δ_B. When the approximation is performed according to the procedure given above, the results are usually of an accuracy comparable to the COSY standard integrator, which is a Runge–Kutta of 8^{th} order with accuracy of about 10^{9-n} for coefficients of order n.

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