Transverse Equilibria in Linear Collider Beam-Beam Collisions

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Abstract

It has been observed in simulations of the beam-beam interaction in linear colliders that a near equilibrium pinched state of the colliding beams develops when the disruption parameter is large ($D \gg 1$)[1]. In this state the beam transverse density distributions are peaked at center, with long tails. We present here an analytical model of the equilibrium approached by the beams, that of a generalized Bennett pinch[2] which develops through collisionless damping due to the strong nonlinearity of the beam-beam interaction. In order to calculate the equilibrium pinched beam size, an estimation of the rms emittance growth is made which takes into account the partial adiabaticity of the collision. This pinched beam size is used to derive the luminosity enhancement factors whose scaling is in agreement with the simulation results for both D and thermal factor $A = \sigma_z/\beta^*$ large, and explains the previously noted cubic relationship between round and flat beam enhancement factors.

Introduction

The calculation of the luminosity enhancement of linear collider beam-beam collisions due to the mutual strong focusing, or disruption, of the beams has been traditionally calculated[1] by use of particle-in-cell computer codes. These numerical calculations solve for electromagnetic fields and the motion of the particles which generate these fields self-consistently. The emergence of near equilibrium pinch-confined transverse beam profiles in the limit that the disruption parameter $D_{x,y} = 2Nr_e\sigma_z/\gamma\sigma_{x,y}(\sigma_x + \sigma_y) \gg 1$ has been noted; in this regime the beam particles undergo multiple betatron oscillations during the collision. It is proposed here that these near equilibrium states are approached through collisionless damping due to mixing and filamentation in phase space, in analogy to a similar phenomena found in self-focusing beams in plasmas[3]. The expected luminosity enhancement obtained in this state is calculated in this paper. Since the approach to this equilibrium entails examining very nonlinear phase space dynamics approximations are necessary, especially with regards to the calculation of the emittance growth induced by filamentation. A model for this emittance growth, based on O. Anderson's theory of space charge induced emittance growth[4], is employed, which then allows a calculation of the luminosity enhancement which is in fairly good agreement with the values obtained by simulation.

Maxwell-Vlasov Equilibria

The equilibria we are proposing to study are of the type known as Maxwell-Vlasov equilibria, which are obtained by looking for a time independent solution of the Vlasov equation describing the beam's transverse phase space, with the forces obtained self-consistently from the Maxwell equations using the beam charge and current profiles. We begin with a flat beam ($\sigma_x \gg \sigma_y$), as these are the simplest, and most likely to be found in a linear collider. For the purpose of calculation the beams are assumed to be uniform in x and z (at least locally), and have identical profiles in y.

The vertical force on an ultra-relativistic particle is thus

$$F_y \simeq -2e|E_y| = -8\pi e^2\Sigma_b \int_0^y Y(y')dy'$$

where Y, normalized by $\int_{-\infty}^{\infty} Y dy = 1$, describes the vertical beam profile, and $\Sigma_b = N/2\pi\sigma_z\sigma_r$ is the beam surface charge density. We look for separable solutions to the time independent Vlasov equation

$$\frac{\partial f}{\partial t} = v_y\frac{\partial f}{\partial y} + F_y\frac{\partial f}{\partial p_y} = 0,$$

where $v_y = p_y/\gamma m$, *i.e.* solutions of the form $f = Y(y)P(p_y)$. This form is in fact approached in true thermal equilibrium. In order for this equilibrium to develop through phase space mixing, more than one nonlinear betatron oscillation must occur during collision, $D_y \simeq 2\pi r_e\Sigma_b\sigma_z^2/\sigma_y \gg 1$.

The solution to the momentum equation obtained in this manner is

$$P(p_y) = \frac{\lambda^2}{\sqrt{2\pi\gamma m}}\exp[-\frac{\lambda^2 p_y^2}{2\gamma m}]$$

which is the Maxwell-Boltzmann form we should expect. The separation constant $\lambda^2 = \gamma m/\sigma_p^2$ is inversely proportional to the temperature of the system. The solution to the corresponding coordinate equation is

$$Y(y) = \frac{\alpha}{2}\mathrm{sech}^2(\alpha y), \quad \alpha = 4\pi e^2 \Sigma_b \lambda^2.$$

This profile is the one-dimensional analogue to the Bennett profile found in cylindrically symmetric Maxwell-Vlasov equilibria. The separation constant λ^2 remains to be calculated in this treatment. As a first attempt, one can use the fact that the distribution function at the origin in phase space is stationary, by symmetry, that is $\partial f/\partial t = 0$ at $(y, p_y) = (0,0)$. Thus $f(0,0)$ is a constant of the motion. Assuming an initial bi-Gaussian distribution in phase space, and equating its peak density in phase space to that of the Bennett-type profile found upon equilibration, we have

$$f(0,0) = Y(0)P(0) = \frac{\Sigma_b}{2\pi\epsilon_n mc} = (e\Sigma_b)^2\lambda^3\sqrt{\frac{2\pi}{\gamma m}}.$$

We thus have, solving directly for α,

$$\alpha = \left[\frac{8\gamma r_e \Sigma_b}{\epsilon_n^2}\right]^{1/3}$$

With this relation we can compare the luminosity that comes about by the transition to a pinch confined Bennett-like state with that of the initial Gaussian beam. At this point, we make allowance for the fact that the beams are not uniform in z and redefine $\Sigma_b \rightarrow \langle\Sigma_b^2\rangle^{1/2} = N/\sqrt{8}\pi\sigma_x\sigma_z$.

Luminosity Enhancement

The luminosity enhancement due to pinch-confinement can be calculated by taking the luminosity integrals of the two cases, assuming $A_y = \sigma_z/\beta_y^* < 1$, the depth of focus effects can be ignored, and $D_x < 1$,

$$\mathcal{L} = \frac{N^2 f_{\mathrm{rep}}}{4\pi\sigma_x\sigma_y} \rightarrow \frac{N^2 f_{\mathrm{rep}}\alpha}{6\sqrt{\pi}\sigma_x}$$

Following this prescription, the luminosity enhancement is

$$H(D_y, A_y) = \frac{4\pi^{1/2}\sigma_y}{3}\left[\frac{\gamma r_e \Sigma_b}{\epsilon_n^2}\right]^{1/3} = \frac{2}{3}(2\pi)^{1/6}\left[\frac{D_y}{A_y^2}\right]^{1/3}$$

This results of the computer simulation of luminosity enhancement by Chen and Yokoya[1] is reprinted in Figure 1. Our expression clearly has too strong a dependence on D_y and A_y to model the simulation results correctly. This is because, even though we have invoked emittance growth (phase space filamentation) as the mechanism behind collisionless approach to equilibrium, we have neglected to calculate this emittance growth.

The emittance growth can be estimated by using a method developed by Anderson to examine space-charge driven emittance growth.[4] We begin by assuming laminar flow of beam particles pinching down under the influence of the opposing beam forces which, ignoring for the

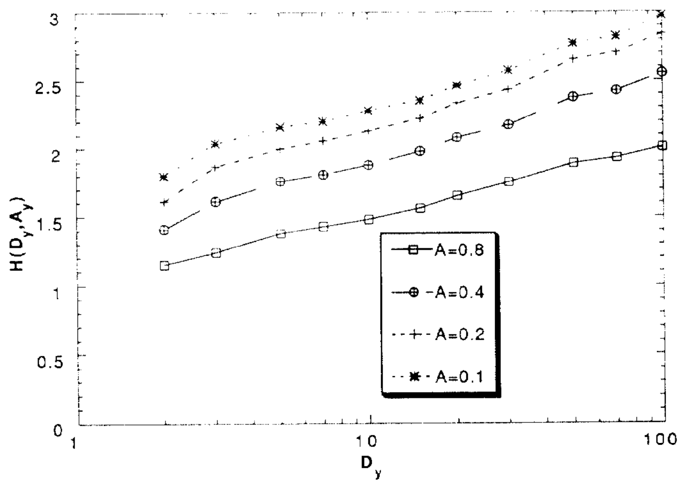

Figure 1: Luminosity enhancement as a function of D_y and A_y, found by computer simulation.

moment the time dependence of the collision, is assumed to be undergoing an identical pinch. The force on the beam particles can then be written in terms of the enclosed beam current, and thus the initial transverse displacement, $F = -8\pi e^2 \Sigma_b G(\xi)$, where $G(\xi) = \int_0^\xi Y(y)dy$ is a constant of the motion under laminar flow conditions. The equation of motion for the beam particles is thus $y'' + K(\xi) = 0$ $(' \equiv d/ds)$, where $K(\xi) = 8\pi r_e \Sigma_b G(\xi)$. For small initial amplitudes $\xi \ll \sigma_y$ we have $K(\xi) = (8\pi r_e n_b/\gamma)\xi \equiv k_\beta^2\xi$, where $n_b = \Sigma_b/\sqrt{2\pi}\sigma_y$ is the beam density on axis. The solution for the small amplitude motion (which for small times is simple harmonic with wavenumber k_β) is thus $y = \xi[1-(k_\beta^2 s^2/2)]$, and all of the small amplitude particles focus at the wave-breaking point $s_{wb} = \sqrt{2}k_\beta^{-1}$. At wave-breaking, which corresponds to one-quarter of a beam oscillation, and after which the laminar flow assumption is violated, the rms emittance has grown explosively and can be calculated (assuming an initially parabolic beam profile of rms beam size σ_y) to be

$$\Delta\epsilon^2 = \langle y\rangle^2\langle y'\rangle^2 - \langle yy'\rangle^2 \simeq \frac{21}{8}\sigma_y^2\left[\frac{r_e N\sigma_y}{\gamma\sigma_x\sigma_z}\right]^2.$$

In fact, not all of this growth can take place, as the initial focusing occurs as the beams see a time-dependent, adiabatically increasing focusing strength. The emittance growth occurs on a length scale of k_β^{-1}, but the beam rethermalizes in a length β_y^* due to the nonlaminar effects of the finite emittance. Thus we must divide our emittance growth factor by $k_\beta\beta_y^* = (2/\pi)^{1/4}\sqrt{D_y/A_y^2}$, and in the spirit of an rms treatment, add it in squares with the initial normalized emittance ϵ_{n0}

$$\epsilon_n^2 \simeq \epsilon_{n0}^2\left[1 + \frac{3}{2}\sqrt{\frac{D_y}{A_y^2}}\right] \simeq \epsilon_{n0}^2\left[1 + \frac{8}{5}k_\beta\beta_y^*\right].$$

The adiabaticity scale length of the collision is σ_z and the initial depth of focus, or rethermalization length, is β_y^*, so

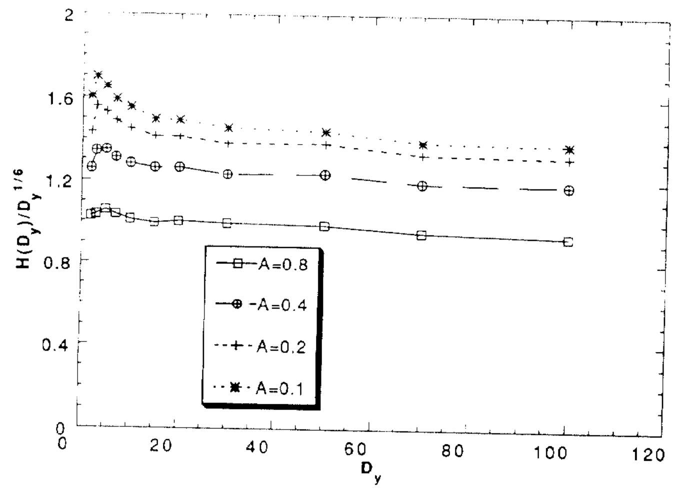

Figure 2: Luminosity enhancement as a function of D_y, multiplied by scaling factor $D_y^{-1/6}$.

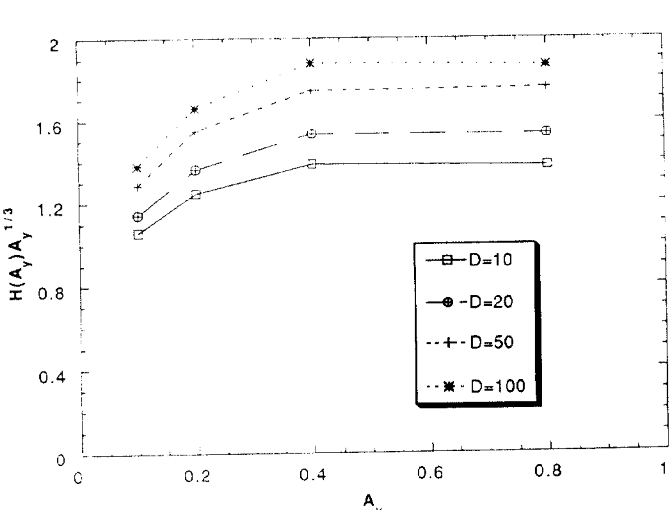

Figure 3: Luminosity enhancement as a function of A_y, multiplied by scaling factor $A_y^{1/3}$.

we must require for this model to hold that A_y be not too much smaller than unity.

Given this emittance growth factor, we can now calculate the luminosity enhancement factor to be

$$H(D_y, A_y) \simeq \frac{2}{3}\left[\frac{\sqrt{2\pi}\frac{D_y}{A_y^2}}{1+\frac{3}{2}\sqrt{\frac{D_y}{A_y^2}}}\right]^{1/3} \simeq \frac{2}{3}\left[\frac{\pi(k_\beta\beta_y^*)^2}{1+\frac{4}{3}k_\beta\beta_y^*}\right]^{1/3}.$$

In the limit of applicability ($D_y \gg 1$, $A_y \lesssim 1$), this relation approaches $H(D_y, A_y) \simeq 0.8(D_y/A_y^2)^{1/6}$. This relation yields scaling which describes the relevant data obtained in simulations quite well, as illustrated in Figs. 2 and 3. Quantitatively, one expects the best agreement when A_y and D_y are largest, and for $D_y = 100$, $A_y = 0.8$ the simulations give $H(D_y, A_y) \simeq 2$, while our scaling gives $H(D_y, A_y) \simeq 1.86$, which is in decent agreement.

The fundamental quantity which governs the luminosity enhancement is evidently $\sqrt{D_y}/A_y \simeq k_\beta\beta_y^*$. The choice of D_y, which is the square of a wave-number, to parameterized this oscillatory interaction is perhaps unfortunate, and is a historical artifact.

Round Beam Enhancement

The emittance growth process for the disruption of round beams, is beyond the scope of this short paper. The results of such a calculation have the same scaling as the flat beam case. We thus confine ourselves to considering, for the purpose of comparison to the flat beam case, the luminosity enhancement in the absence of emittance growth.

Following the treatment in Ref. [3], the Bennett profile of the pinch-confined beam system is

$$R(r) \sim \left[1+(r/a)^2\right]^{-2},$$

where $a^2 = 2\epsilon_n^2/\gamma\nu$, where the Budker parameter $\nu = Nr_e/2\sqrt{pi}\sigma_z$ (we have again taken the rms value of the charge density). The luminosity enhancement obtained in this case (*not including emittance growth*) is

$$H(D, A) = \frac{\sigma_r^2}{3a^2} = \frac{D}{12\sqrt{\pi}A^2} \sim \frac{D}{A^2}.$$

By comparing this to the equivalent flat beam expression, we see that

$$H_{\text{flat}} \sim H_{\text{round}}^{1/3},$$

a relationship which has been previously deduced from the simulation data[1]. This is further confirmation that our model incorporates much of the relevant physics of beam-beam disruption.

References

[1] P. Chen and K. Yokoya, *Phys. Rev. D* **38**, 987 (1988) and P. Chen, in *Physics of Particle Accelerators,* Ed. M. Month and M. Dienes, 633 (AIP, New York, 1989).

[2] W. H. Bennett, *Phys. Rev.* **45**, 890 (1934), and *Phys. Rev.* **98**, 1584 (1955).

[3] J. B. Rosenzweig, P. Schoessow, B. Cole, W. Gai, C. Ho, R. Konecny, S. Mtingwa, J. Norem, M. Rosing, and J. Simpson, *Phys. Fluids B* **2**, 1376 (1990).

[4] O.A. Anderson, *Particle Accelerators* **21**, 197 (1987).