Modelling of Space Charge Effects in the CERN Proton Synchrotron

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Abstract

The performance of the CERN Proton Synchrotron (PS) at injection is limited by resonance effects caused by space charge tune spreads.

This paper describes a new computer code for calculating the incoherent tune spreads and the beam envelopes in linear coupled synchrotron lattices in the presence of transverse and longitudinal space charge fields. This work is based on an existing theory devised at DESY using the six-dimensional phase space formalism. It has been extended to deal with the nonuniform charge density within the bunches. The application to high intensity beams at present in operation in the PS is discussed.

I. INTRODUCTION

The understanding of the space charge effects is of primary importance for low energy circular accelerators and storage rings. Theoretical models which permit reliable numerical simulation are desirable to analyze the beam behaviour.

In this paper the original formalism developed at DESY [1, 2] is extended by taking into account nonuniform charge density within bunches. Throughout this text the same variables will be used as those in [2], i.e. $x, z, \sigma = s - \beta ct, \eta = \Delta E/E$, where x, z describe the transverse betatron oscillations, σ, η the synchrotron oscillations in the longitudinal plane, s the arc length of the reference orbit, and β, E the velocity in units of c and the total energy of the reference particle respectively.

II. EQUATIONS OF MOTION

In a matrix form the equation of motion of a particle of rest mass m_0 and charge e in a linear lattice in presence of space charge effects and coupling writes

$$\vec{y}\,' = \underline{A}(s, \vec{y})\,\vec{y} \tag{1}$$

with $\vec{y}^{\,t} = (x, p_x, z, p_z, \sigma, \eta)$ where

$$p_x = \beta^2 x' - H(s)z \tag{2}$$

$$p_z = \beta^2 z' + H(s)x \tag{3}$$

and

$$\begin{aligned}
A_{1,2} &= A_{3,4} = \beta^{-2}\\
A_{1,3} &= A_{2,4} = H\\
A_{2,1} &= -\beta^2\left(K_x^2 + g + H^2 - F_{xx}(\vec{y})\right)\\
A_{2,3} &= A_{4,1} = \beta^2\left(N + F_{xz}(\vec{y})\right)\\
A_{2,6} &= -A_{5,1} = K_x\\
A_{3,1} &= A_{4,2} = -H\\
A_{4,3} &= -\beta^2\left(K_z^2 - g + H^2 - F_{zz}(\vec{y})\right)\\
A_{4,6} &= -A_{5,3} = K_z\\
A_{5,6} &= \beta^{-2}\gamma^{-2}\\
A_{6,5} &= \frac{2\pi h e\hat{V}}{E_0 L}\cos\varphi\sum_{\nu=1}^{m}\delta(s - s_\nu) + F_\sigma(\vec{y})\\
A_{i,j} &= 0 \;\; otherwise
\end{aligned} \tag{4}$$

with $\gamma = 1/\sqrt{1-\beta^2}$. Here $g(s)$ is the quadrupole strength, $N(s), H(s)$ are the skew quadrupole components and solenoid fields, $K_x(s), K_z(s)$ are the curvatures in the x and y directions, and m is the number of cavities (assumed point like at $s = s_\nu$). The peak electric field in a cavity is $\hat{V}$, $\varphi = 0, \pi$ (no acceleration), h is the harmonic number and L is the length of the reference orbit [2].

The space charge forces depend on $\vec{y}$ for nonuniform charge distribution. The terms $F_{xx}(\vec{y})$, $F_{xz}(\vec{y})$, $F_{zz}(\vec{y})$ and $F_\sigma(\vec{y})$ describe the self field space charge forces and will be defined later.

III. SPACE CHARGE FORCES

Bunched beams of ellipsoidal shape with nonuniform charge density in the ellipsoid are considered. The three-dimensional model of the bunch is a nest of N concentric ellipsoids (numbered from the inner towards the outer) whose shells (each of sufficiently small thickness) are assumed to have a uniform charge density.

Synchro-betatron coupling other than by space charge forces will be neglected. Thus the twist angles $\theta_{x\sigma}$ and $\theta_{z\sigma}$ of the bunch with respect to the σ axis in the x–σ and z–σ planes can be ignored.

The force acting on a particle in the nth layer can be obtained from the expressions for the potential of homogeneous ellipsoids. Assuming that the longitudinal dimension is much greater than the transverse dimensions ($\gamma E_\sigma \gg E_1$, E_2), the components of the transverse space charge force in the nth layer in a rotated coordinate system $(\tilde{x}, \tilde{z}, \sigma)$ parallel to the half axes $E_{1,n}$, $E_{2,n}$, $E_{\sigma,n}$ of the nested ellipsoids are (in the laboratory system)

$$F^n_{\tilde{x}} = \frac{e}{\gamma\epsilon_0} I^n_1(\tilde{x}, \tilde{z}, \sigma)\tilde{x} \tag{5}$$

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$$F_\sigma^n = \frac{e\gamma}{\epsilon_0} I_3^n(\tilde{x},\tilde{z},\sigma)\sigma \tag{6}$$

with the approximation

$$I_1^n(\tilde{x},\tilde{z},\sigma) = \sum_{i=n+1}^{N} \rho_i \left(\frac{E_{2,i}}{E_{1,i}+E_{2,i}} - \frac{E_{2,i-1}}{E_{1,i-1}+E_{2,i-1}} \right) + \sum_{i=1}^{n-1} (\rho_i - \rho_{i+1}) \frac{E_{1,i}\,E_{2,i}}{E_{1,i}^2 - E_{2,i}^2} \left(1 - \sqrt{\frac{k_{i,n}+E_{2,i}^2}{k_{i,n}+E_{1,i}^2}} \right) + \rho_n \frac{E_{2,n}}{E_{1,n}+E_{2,n}} \tag{7}$$

Similar expression hold for $F_{\tilde{z}}^n$ and I_2^n provided that $\tilde{z}$ replaces $\tilde{x}$ in Eq. 5 and $E_{1,i}$ is interchanged with $E_{2,i}$ in Eq. 7. Further

$$I_3^n(\tilde{x},\tilde{z},\sigma) = \sum_{i=n+1}^{N} \rho_i \left(\frac{E_{1,i}\,E_{2,i}}{E_{\sigma,i}^2} P_i - \frac{E_{1,i-1}\,E_{2,i-1}}{E_{\sigma,i-1}^2} P_{i-1} \right) + \sum_{i=1}^{n-1} (\rho_i - \rho_{i+1}) \frac{E_{1,i}\,E_{2,i}}{E_{\sigma,i}^2} (P_i + Q_{i,n}) + \rho_n \frac{E_{1,n}\,E_{2,n}}{E_{\sigma,n}^2} P_n \tag{8}$$

with

$$P_i = \ln \left(\frac{E_{\sigma,i} + \sqrt{E_{1,i}^2 - E_{2,i}^2 + E_{\sigma,i}^2}}{E_{1,i}+E_{2,i}} \right) \tag{9}$$

$$Q_{i,n} = \ln \left(\frac{\sqrt{k_{i,n}+E_{1,i}^2} - \sqrt{k_{i,n}+E_{2,i}^2}}{E_{1,i}-E_{2,i}} \right) \tag{10}$$

where ρ_i is the constant charge density inside the ith layer and $k_{i,n}$ is the largest root of the equation

$$\frac{\tilde{x}^2}{E_{1,i}^2 + k_{i,n}} + \frac{\tilde{z}^2}{E_{2,i}^2 + k_{i,n}} + \frac{\sigma^2}{E_{\sigma,i}^2 + k_{i,n}} = 1 \tag{11}$$

which depends on the location $(\tilde{x},\tilde{z},\sigma)$ in the layer. An approximate mean value for $k_{i,n}$ is obtained by taking the average of Eq. 11 over the volume of the shell. This approximation means that I_1^n, I_2^n, I_3^n keep constant values within the nth shell. Thus the nonlinear character of space charge forces will be approximated by piecewise linear functions in the N ellipsoidal level layers.

The space charge force components with respect to the (x,z) axes become, using Eqs. 5–6

$$F_x^n = m_0 c^2 \beta^2 \gamma \, (F_{xx}^n x + F_{xz}^n z) \tag{12}$$

$$F_z^n = m_0 c^2 \beta^2 \gamma \, (F_{xz}^n x + F_{zz}^n z) \tag{13}$$

with

$$F_{xx}^n = \frac{e}{\epsilon_0 m_0 c^2 \beta^2 \gamma^2} \left(I_1^n \cos^2 \theta_{xz}^n + I_2^n \sin^2 \theta_{xz}^n \right) \tag{14}$$

$$F_{zz}^n = \frac{e}{\epsilon_0 m_0 c^2 \beta^2 \gamma^2} \left(I_1^n \sin^2 \theta_{xz}^n + I_2^n \cos^2 \theta_{xz}^n \right) \tag{15}$$

$$F_{xz}^n = \frac{e}{\epsilon_0 m_0 c^2 \beta^2 \gamma^2} (I_1^n - I_2^n) \sin\theta_{xz}^n \, \cos\theta_{xz}^n \tag{16}$$

where θ_{xz}^n is the twist angle of the nth bunch layer with respect to the x axis in the x-z plane.

Hence, the space charge terms $F_{xx}(\vec{y})$, $F_{xz}(\vec{y})$, $F_{zz}(\vec{y})$ and $F_\sigma(\vec{y})$ in Eq. 4 are constituted of the piecewise approximations F_{xx}^n, F_{xz}^n, F_{zz}^n and F_σ^n, for $n = 1 \ldots N$.

IV. Solution of the Equations

For a particle with coordinates (x,z,σ) lying in the nth ellipsoidal shell, Eq. 1 can be linearized so that its solution can be written in the form

$$\vec{y}^{\,n}(s) = \underline{M}^n(s,s_0)\,\vec{y}^{\,n}(s_0) \tag{17}$$

where $\underline{M}^n(s,s_0)$ is the transfer matrix and $\vec{y}^{\,n}(s_0)$ is an initial vector.

Although a rigorous symplectic thin lens approximation for the transfer matrix has been established [1, 2], the simpler commonly used thin lens linear approximation (non symplectic) is considered here because it yields in practice similar results

$$\underline{M}^n(s+\Delta s, s) = \underline{I} + \Delta s\, \underline{A}^n(s) \tag{18}$$

Here $\underline{A}^n(s)$ is the matrix $\underline{A}(s)$ in which the space charge terms are replaced by their piecewise approximation.

Let $\vec{y}_k^{\,n}(s_0)$ ($k = 1 \ldots 6$) be linearly independent vectors. Then the following vector spans an hyperellipsoid in the six dimensional phase space x–p_x–z–p_z–σ–η by varying the angles $\varphi, \chi, \delta_1, \delta_2, \delta_3$ [2]

$$\begin{aligned} \vec{y}^{\,n}(s_0;\varphi,\chi,\delta_1,\delta_2,\delta_3) = \\ \cos\varphi\cos\chi\,(\vec{y}_1^{\,n}(s_0)\cos\delta_1 + \vec{y}_2^{\,n}(s_0)\sin\delta_1) + \\ \cos\varphi\cos\chi\,(\vec{y}_3^{\,n}(s_0)\cos\delta_2 + \vec{y}_4^{\,n}(s_0)\sin\delta_2) + \\ \sin\varphi\,(\vec{y}_5^{\,n}(s_0)\cos\delta_3 + \vec{y}_6^{\,n}(s_0)\sin\delta_3) \end{aligned} \tag{19}$$

The projections of the hyperellipsoid onto the x–z, x–σ, z–σ and x–p_x, z–p_z, σ–η planes yield the bunch cross sections for the nth layer (i.e. the half axes of the ellipses obtained by projection of the hyperellipsoid) [2]. Hence the bunch envelope comprising, say, 95% of particles, will be given by the cross sections of the $n_{0.95}$th layer.

V. Initial Conditions and Charge Density for the Hyperellipsoid

The transfer matrix over one machine turn must be periodic. This is the case if the hyperellipsoid spanned by (19) recovers its original shape after one revolution, i.e. if $\vec{y}^{\,n}(s_0;\varphi,\chi,\delta_1,\delta_2,\delta_3)$ transforms after one turn into $\vec{y}^{\,n}(s_0;\varphi,\chi,\delta_1 - 2\pi Q_1^n, \delta_2 - 2\pi Q_2^n, \delta_3 - 2\pi Q_3^n)$. Here $Q_{1,2}^n$ and Q_3^n characterize oscillation modes which no longer correspond to pure horizontal, vertical and longitudinal motion. However they are still identified with the transverse machine tunes and the synchrotron tune respectively. The periodic condition involves the following computational steps ($n = 1 \ldots N$, $j = 1, 2, 3$):

- Find the spectrum of the revolution matrix (eigenvalues must be complex of module unity)

$$\underline{M}^n(s_0 + L, s_0)\, \vec{v}_j^n(s_0) = e^{-2i\pi Q_j^n}\, \vec{v}_j^n(s_0) \qquad (20)$$

- Normalize the vectors $\vec{v}_j^n(s_0)$ and form

$$(\vec{v}_j^n)^+(s_0)\, \underline{S}\, \vec{v}_j^n(s_0) = 2i \qquad (21)$$

$$\sqrt{\epsilon_j^n(s_0)}\, \vec{v}_j^n(s_0) = \vec{y}_{2j-1}^n(s_0) - i\vec{y}_{2j}^n(s_0) \qquad (22)$$

where $\underline{S}$ stands for the *unit symplectic* matrix.

The quantities $\vec{v}_j^n(s_0)$, Q_j^n and ϵ_j^n must be computed iteratively because the revolution matrix depends on its eigenvalues and eigenvectors. In the first iteration the space charge forces are ignored and the values $\epsilon_{1,2,3}^n$ are derived from a system of three nonlinear algebraic equations defining the boundary of the beam cross sections at injection

$$\pi\epsilon_{x,z,\sigma} = f_{x,z,\sigma}(\vec{y}_1^N(s_0), \ldots, \vec{y}_6^N(s_0), \epsilon_1^N, \epsilon_2^N, \epsilon_3^N) \qquad (23)$$

and for the subsequent ellipsoidal level layers

$$\epsilon_j^n = r_n^2\, \epsilon_j^N \qquad (24)$$

where $\pi\epsilon_x$, $\pi\epsilon_z$ and $\pi\epsilon_\sigma$ are the areas of the elliptical projection of (19) onto the corresponding phase planes, and r_n is obtained from the initial charge distribution. Eqs. 23 are solved for the case of beams injected with no twist.

In the limiting case of negligible space charge forces and no coupling between the synchrotron and betatron oscillations and between the betatron oscillations themselves, $\epsilon_{1,2}^N$ and ϵ_3^N are approximately equal to the transverse and the longitudinal emittances respectively.

The initial charge distribution $\rho(x, p_x, z, p_z, \sigma, \eta)$ occupies the six dimensional ellipsoid spanned by (19). Assuming a charge density with ellipsoidal symmetry, this hyperellipsoid can be transformed into an hypersphere so that charge distribution $\rho(\tilde{r})$ only depends on the radius $\tilde{r} = \sqrt{r^2 + r'^2}$, with $r^2 = \bar{x}^2 + \bar{z}^2 + \bar{\sigma}^2$ and $r'^2 = \bar{p_x}^2 + \bar{p_z}^2 + \bar{\eta}^2$ (the *bar* means that the values are normalized) [3]. Hence, since the charge density is constant on any hypersphere of radius $\tilde{r}$, the distribution $\rho(r)$ in the real space is obtained by integration of $\rho(r, r')$ over r'.

Thus a nest of N concentric spheres in the $\bar{x}$–$\bar{z}$–$\bar{\sigma}$ space (of density $\rho(r)$) can be derived from a family of N circles of radius $\tilde{r}_n$ (of density $\rho(\tilde{r})$) by projection onto the r axis, yielding a family of segments of length r_n (the outer circle has a radius $\tilde{r}_N = 1$). If two consecutive circles have radii $\tilde{r}_n$ and $\tilde{r}_{n+1}$ close to each other, the local charge density $\rho_n(\tilde{r})$, and so $\rho_n(r)$, can be assumed uniform.

Denormalizing the six dimensional phase space variables yields a nest of concentric ellipsoids which can be used as a model for the bunch. A Gaussian distribution has been chosen to represent the charge distribution $\rho(x, p_x, z, p_z, \sigma, \eta)$, yielding by projection a Gaussian distribution in the real space. Uniform charge density in the real space is also considered as a limiting case [2].

VI. Conclusion

A simulation code has been written and was used to study the effects of space charge self fields at 1 GeV (kinetic energy) injection into the PS for the present high intensity beam delivered to the SPS (20 bunches, each of 10^{12} protons, 55 ns long (at 4σ), 1.3 MeV half energy spread and normalized emittances $\epsilon_x^* = 50\mu m$, $\epsilon_z^* = 25\mu m$ (at 2σ)).

Both Gaussian and uniform charge distributions have been used for the simulation. The particle bunch was sliced into 8 ellipsoidal layers. The tune values (without space charge) are $Q_{0,1} = 6.217$ and $Q_{0,2} = 6.361$. Fig. 1 shows the tune spread in the PS for a Gaussian charge density within the bunch. When a uniform charge density is considered the calculated tune shifts are $\Delta Q_1 = -0.147$ and $\Delta Q_2 = -0.214$. The calculated tune spread has no practical effect on beam performance as no significant beam losses or beam blow-up have ever been observed for this intensity.

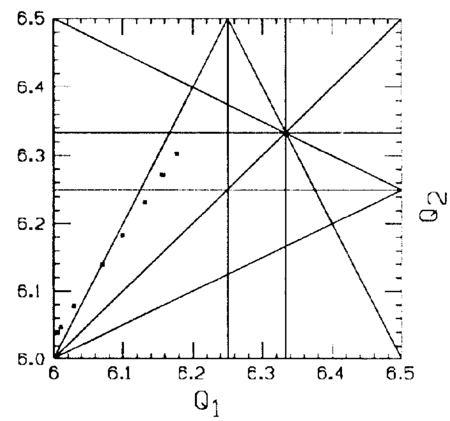

Figure 1: Distribution of incoherent tune at 1 GeV in the PS

References

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