On Dynamic Aperture

Zohreh Parsa
Brookhaven National Laboratory
Upton, NY 11973

Abstract

Nonlinear perturbation can alter the beam behavior in dynamical systems. For example, nonlinear fields due to sextupoles or octupoles can perturb the motion of the beam of particles in accelerators. If the perturbation is increased, at some point, the motion of the beam become unstable and chaotic. In the Superconducting Super Collider (SSC) a proton would interact 10^{12} times (in $\sim 10^9$ revolutions) with the local nonlinear magnetic fields which can cause the deviations, of the beams from the designed orbit, and lead to loss of particles. We present an analytical method to study the beam stability in large accelerators such as SSC, (and an alternate to using "kick" approximation (particle tracking) over large time intervals). This includes stability factor "f" which determines the convergence of the superconvergent perturbation theory, and the behavior of the system, e.g., stable if $f > 1$, and unstable if $f < 1$. The critical point where the system makes the transition from stable ($F > 1$) to chaotic ($f < 1$) behavior corresponds to $f = 1$, the point at which the size of the beam is defined as the dynamic aperture in accelerators.

I. INTRODUCTION

A method for determining the dynamic aperture in accelerators is discussed, which corresponds to finding the transition point from a stable to chaotic behavior. When the stability factor f, which determines the convergence of the superconvergent perturbation theory, (a method used to study the stability of dynamical systems, e.g. used in proof of K.A.M. Theorem), is equal to 1.

II. FORMALISM

According to K.A.M. Theorem [1], if a dynamical system described by the Hamiltonian

$$H = T + V_N \tag{1}$$

has no degenerecies in its frequencies, (there exist an $m > 0$ such that $|V_N| < m$), then for a small perturbation (V_N), the motion is not stochastic and can be approximated by analytic functions. That is, under a small analytic perturbation the majority of the invariant tori do not collapse but are only slightly deformed [1]. Our aim is to find m, the upper limit on the strength of this perturbation, such that K.A.M. theorem still holds.

In a case where the KAM boundary between stable and stochastic motion might be at a finite amplitude but the perturbation series does not converge to analytic functions for the constants of motion even below the limit (e.g. where there is a stability bound beyond which the phase plot exhibits island structure, although the motion is not stochastic), another test (e.g. overlap criteria [2–9]), should be used for confirmation.

Consider a Hamiltonian:

$$H_l = T_l\left(\overrightarrow{J}_l, s\right) + V_l\left(\overrightarrow{J}_l, \overrightarrow{\Phi}_l, s\right) \tag{2}$$

where T is the kinetic energy, V is the perturbed potential, l = integer, J_l and Φ_l are l dimensional (action and angle conjugate) vectors, such that $H_0 = H, T_0(\overrightarrow{J}_0, s) = T(\overrightarrow{J}, s)$, $V_0(\overrightarrow{J}_0, \overrightarrow{\Phi}_0, s) = V_N(\overrightarrow{J}, \overrightarrow{\Phi}, s)$, $\overrightarrow{J}_0 = \overrightarrow{J}$ and using the generating function

$$F_l = \overrightarrow{\Phi}_l \bullet \overrightarrow{J}_{l+1} + G_l\left(\overrightarrow{J}_{l+1}, \overrightarrow{\Phi}_l, s\right) \tag{3}$$

from H_l, $\overrightarrow{\Phi}_0 = \overrightarrow{\Phi}$. H_{l+1} is found and we obtain:

$$H_{l+1} = T_{l+1}\left(\overrightarrow{J}_{l+1}, s\right) + V_{l+1}\left(\overrightarrow{J}_{l+1}, \overrightarrow{\Phi}_{l+1}, s\right) \tag{4}$$

where

$$T_{l+1} = T_l\left(\overrightarrow{J}_{l+1}, s\right) + A_l\left(\overrightarrow{J}_{l+1}, s\right) \tag{5}$$

and

$$\begin{aligned}V_{l+1} = &\left[T_l\left(\overrightarrow{J}_{l+1} + \partial_{\overrightarrow{\Phi}_l} G_l, s\right) - T_l\left(\overrightarrow{J}_{l+1}, s\right)\right.\\ &\left. - \partial_{\overrightarrow{J}_{l+1}} T_l\left(\overrightarrow{J}_{l+1}, s\right) \bullet \partial_{\overrightarrow{\Phi}_l} G_l\right]\\ &+ \left[V_l\left(\overrightarrow{J}_{l+1} + \partial_{\overrightarrow{\Phi}_l} G_l, \overrightarrow{\Phi}_l, s\right) - V_l\left(\overrightarrow{J}_{l+1}, \overrightarrow{\Phi}_l, s\right)\right]\\ &+ \left[V_l\left(\overrightarrow{J}_{l+1}, \overrightarrow{\Phi}_l, s\right) - \left\{V_l\left(\overrightarrow{J}_{l+1}, \overrightarrow{\Phi}_l, s\right)\right\}_{2^l N}\right]\\ &- A_l\left(\overrightarrow{J}_{l+1}, s\right)\end{aligned} \tag{6}$$

0-7803-0135-8/91$01.00 ©IEEE

The term $A_l(\vec{J}_{l+1}, s)$ is included in the above two equations to eliminate the nonresonant term from V_{l+1} in Eq. (6). The Fourier expansion of V_l about $\vec{\Phi}_l$ is

$$\left\{V_l\left(\vec{J}_{l+1}, \vec{\Phi}_l, s\right)\right\}_{2^l N} = \sum_{0<|\vec{m}|\le 2^l N} v_{l\vec{m}}\left(\vec{J}_{l+1}, 2\right) e^{i\vec{m}\bullet\vec{\Phi}_l} \tag{7}$$

where $\vec{m}$ is an n-dimensional vector with integer components and $v_{l\vec{m}}$ are the Fourier amplitude of V_l; and we expect that the potential term V_l goes to zero in the limit $l \to \infty$ if there exist "n" invariants i.e.,

$$\lim_{l\to\infty} V_l\left(\vec{J}_l, \vec{\Phi}_l, s\right) = 0. \tag{8}$$

Next we express the G_l term in the generating function as:

$$G_l = \sum_{\vec{m}} g_{l\vec{m}}\left(\vec{J}_{l+1}, s\right) e^{i\vec{m}\bullet\vec{\Phi}_l} \tag{9}$$

and define the phase advance $\vec{\Psi}_l$ and tune $\vec{\nu}_l$ as:

$$\vec{\Psi}_l\left(\vec{J}_{l+1}, s\right) \equiv \partial_{\vec{J}_{l+1}} T_l\left(\vec{J}_{l+1}, s\right) \tag{10a}$$

$$\vec{\nu}_l\left(\vec{J}_{l+1}\right) \equiv \frac{1}{2\pi}\int_0^L \vec{\Psi}_l\left(\vec{J}_{l+1}, s\right) ds \tag{10b}$$

where Ψ is periodic in s, with L the length of the period (e.g. a circumference of an accelerator) i.e. $\vec{\Psi}_l(\vec{J}_{l+1}, s) = \vec{\Psi}_l(\vec{J}_{l+1}, s+L)$, and obtain

$$G_l\left(\vec{J}_{l+1}, \vec{\Phi}_l, s\right) = \sum_{\vec{m}} \frac{1}{2\sin\pi\left(\vec{m}\bullet\vec{\nu}_l\left(\vec{J}_{l+1}\right)\right)} \int_s^{s+L} v_{l\vec{m}}\left(\vec{J}_{l+1}, t\right) \exp\left\{i\vec{m}\bullet\left[\vec{\Phi}_l + \vec{\Psi}_l\left(\vec{J}_{l+1}, t\right) - \vec{\Psi}_l\left(\vec{J}_{l+1}, s\right) - \pi\vec{\nu}_l\left(\vec{J}_{l+1}\right)\right]\right\} dt \tag{11}$$

from this we get a bound on V_{l+1} given the magnitude of v_l using Eq. (6) and expanding the terms in V_{l+1} (which depends on $\vec{J}_{l+1} + \partial_{\vec{\Phi}_l} G_l$) in the Taylor series about $\vec{J}_{l+1}$:

$$v_{l+1}\left(\vec{J}_{l+1}, \vec{\Phi}_l, s\right) \equiv \tfrac{1}{2}\partial_{\vec{\Phi}_l} G_l \bullet \partial_{\vec{J}_{l+1}} \vec{\Psi}_l\left(\vec{J}_{l+1}, s\right) \bullet \partial_{\vec{\Phi}_l} G_l + \partial_{\vec{J}_{l+1}} V_l\left(\vec{J}_{l+1}, \vec{\Phi}_l, s\right) \bullet \partial_{\vec{\Phi}_l} G_l + 0\left(v^3\right) + \left[V_l\left(\vec{J}_{l+1}, \vec{\Phi}_l, s\right) - \left\{V_l\left(\vec{J}_{l+1}, \vec{\Phi}_l, s\right)\right\}_{2^l N}\right] \tag{12}$$

Substituting the generating function G_l into (12) and after changing the angle variables from Q_l to Q_{l+1}: (for $|\vec{m}| < 2^l N$), and using triangle inequality, defining

$$D_{l\vec{m}}\left(\vec{J}_{l+1}\right) \equiv \sup_{s\to s+L} \left|v_{l\vec{m}}\left(\vec{J}_{l+1}, s\right)\right| \tag{13}$$

and the $M_l(\vec{J})$ as the upper bound of the potential

$$M_l\left(\vec{J}_l\right) \equiv \sum_{\vec{m}} D_{l\vec{m}}\left(\vec{J}_l\right) \tag{14}$$

we find the relation between $M_l(\vec{J}_l)$ and $M_{l+1}(\vec{J}_{l+1})$:

$$M_{l+1} \le \frac{n^2}{4} L\left(\frac{A_l L}{2}\mu_l\xi_l + \frac{1}{|J_{l+1}|}\lambda_l\zeta_l\right) M_l^2 \tag{15}$$

with

$$\xi_l\left(J_{l+1}\right) \equiv \sum_{\vec{m}_1} \frac{|\vec{m}_1|}{|\sin\pi\vec{m}_1 \bullet \vec{\nu}_l|} \quad \text{and} \quad \zeta_l \equiv \sum_{|m_1|} |\vec{m}_1| \tag{16}$$

where $\mu_l(\vec{J}_{l+1})$ and $\lambda_l(\vec{J}_{l+1})$ are $\approx$ 1/the number of terms in each sum respectively. Raising Eq. (15) to (2^{-l-1}) power and defining $F_l \equiv m_l^{2^{-l}}$ (i.e. translating Eq. (15) to a linear difference equation) we obtain

$$F_{l+1} \le \left[\frac{n^2}{4} L\mu_l\xi_l\left(\frac{A_l L}{2}\mu_l\xi_l + \frac{1}{|\vec{J}_{l+1}|}\lambda_l\zeta_l\right)\right]^{2^{-l-1}} F_l \tag{17}$$

defining

$$F_{l+1} \equiv b\left(l, J_{l+1}\right) F_l \tag{18}$$

where

$$b\left(l, \vec{J}_{l+1}\right) \le \left[\frac{n^2}{4} L\mu_l\xi_l\left(\frac{A_l L}{2}\mu_l\xi_l + \frac{1}{|\vec{J}_{l+1}|}\lambda_l\zeta_l\right)\right]^{2^{-l-1}} \tag{19}$$

we obtain F_l as the solution to Eq. (18) to be:

$$F = \left[\prod_{k=0}^{l} b\left(k, \vec{J}_{k+1}\right)\right] F_0 \tag{20}$$

From this equation we determine the stability of the dynamical system, e.g. when the $f \equiv \lim_{l\to\infty} F_l = 1$ (which corresponds to $\lim_{l\to\infty} M_l = 0$), the dynamical system remains stable, (this is similar to the "root test" for an infinite series) [3]. Thus we can define a stability factor

$$f \equiv \lim_{l\to\infty} F_l = \left[\prod_{k=0} b\left(k, \vec{J}_{k+1}\right)\right] M_0 \qquad (21)$$

such that as long as $f < 1$ the dynamic system remains stable, becomes chaotic when $f > 1$ (e.g. for large perturbation), and makes the transition from stable to unstable behavior for $f = 1$ (or could occur at $\sim f = 1^-$ or $f = 1^+$). The dynamical aperture in accelerators, is defined as the size of the beam at which the particles become unstable. This corresponds to $f = 1$, which is the transition point from stable to chaotic behavior. Using Eq. (19) one can estimate the dynamic aperture for accelerators. For example the presence of the large β functions, at the SSC (Superconducting Super Collider) quadrupole triplets in the low β insertions, leads to a large perturbing potential (V_N, due to multipoles, etc.) that adds a large contribution to M_0 in Eq. (21), and greatly reduces the dynamic aperture in that accelerator. Thus requiring a smaller size beam in order to avoid instability (for $f = 1$). This suggests, that by reducing the multipole contributions in the SSC triplet quadrupole magnets, the dynamic aperture may be improved.

III. SUMMARY

Using a full superconvergent perturbation theory, one can calculate ($b(l, J_{l+1})$ exactly resulting in) a more precise estimate of the dynamic aperture (the transition point from stable to stochastic motions with $f = 1$), which is a useful analytical tool in accelerator design.

IV. REFERENCES

1. V.I. Arnold, Russian Math. Surveys, vol. 18, 1963.
2. Z. Parsa, "Accelerator dynamics and beam aperture", BNL 29194, 1986.
3. Z. Parsa, S. Teikian and E. Courant, "Second order perturbation for accelerators", Particle Acc., vol. 22, pp. 205–230, 1987.
4. Z. Parsa, S. Tepikian, (unpublished).
5. Z. Parsa, "Analytical method for obtaining the variations of the beam emittance, particle action and linear aperture in accelerators", *Proc. of the 1986 Summer Study on the Physics of the SSC*, Snowmass, Co., Edited by R. Donaldson and J. Marx, June 23–July 11, 1986, p. 573, and BNL Report #38735.
6. SSC Central Design Group, "SSC conceptual design", SSC-SR-2020, March 1986.
7. R.C. Buck, *Advanced Calculus*, McGraw-Hill Inc. 1978.
8. B.V. Chirikov, Phys. Rept. 52, 1979.
9. Z. Parsa, "Analytical method for treatment of nonlinear resonances in accelerators", *Proc. of the 1986 Summer Study on the Physics of SSC*, Snowmass, Co., Edited by R. Donaldson and J. Marx, June 23–July 11, 1986, p. 579, and BNL Report #38734.
10. J.M. Greene, J. Math. Phys., vol. 20, pp. 1183–1201, 1979.

* We thank Ms. Fern Simes for assisting with the typing of the manuscript.