

Resonance Seeding of Stability Boundaries in Two and Four Dimensions

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1 Introduction.

“Resonance seeding” refers to the hypothesis that the stochastic layer delineating the dynamic aperture of a Hamiltonian system grows out of separatrices generated by very low order resonances.[2, 3] This is a *physics* hypothesis and should not be interpreted as arising from any particular technique for writing perturbative expansions, such as the ones developed by Deprit, Dragt, or Forest. Although analytic representations of the resonances are indeed obtained via perturbation theory, existence of the separatrices and the validity (or otherwise) of resonance seeding are separate from it. We shall describe some of the evidence supporting this idea in two and four dimensions.

2 Two-dimensional maps

With suitable definitions of transverse position and “momentum,” the Poincaré map representing insertion of a thin sextupole into a storage ring can be written,

$$\begin{pmatrix} x \\ p \end{pmatrix} \leftarrow \begin{pmatrix} \cos 2\pi\nu & \sin 2\pi\nu \\ -\sin 2\pi\nu & \cos 2\pi\nu \end{pmatrix} \begin{pmatrix} x \\ p - \lambda x^2 \end{pmatrix}, \quad (1)$$

where $\lambda = -e\beta B''l/2p_3$, e being the charge on a proton (the assumed probe), p_3 its longitudinal momentum, $B''l$ the integrated second derivative of the sextupole field, and β the envelope function evaluated at the position of the sextupole. We can set $\lambda \equiv 1$ without loss of generality by rescaling, $x \rightarrow x/\lambda$ and $p \rightarrow p/\lambda$. (In nonlinear circles, this is called the Hénon map. See Henon, M. Quart. App. Math. 27,291(1969).)

Comparison of this map with perturbation theory calculations was reported in [2]. We have studied this mapping in the tune range $0 < \nu < \frac{1}{2}$; Figures 1a and 2a illustrate a few orbits at the tunes $\nu = 0.1$, $\nu = 0.29$ respectively. The general features in these drawings are not surprising: (i) near the origin there are smooth (on the scale of the observations) KAM tori; (ii) as one gets farther in phase space a structure of islands and sub-islands develops; (iii) which

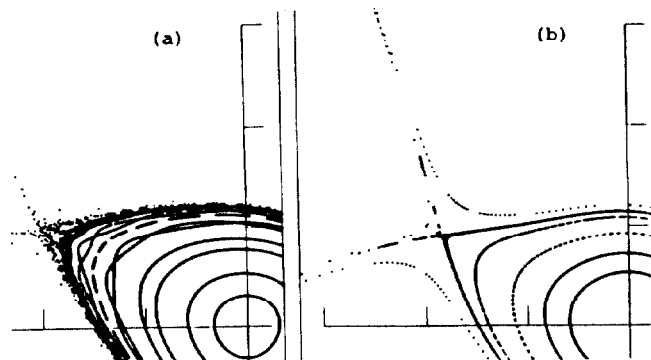


Figure 1: (a) Orbits of the sextupole mapping for $\nu = 0.15$. (b) Second order perturbation theoretic calculation of the stability boundary. The tick marks on the axes are separated by 0.5.

finally breaks into a chaotic sea, the “stability boundary” marking the system’s dynamic aperture.

Figures 1b and 2b illustrate calculations done by applying second order ($O(\lambda^2)$) perturbation theory to the map Eq.(1). The dynamics in Figure 1 are dominated by a first order integer resonance, which is put explicitly into a resonant normal form Hamiltonian. With the appropriate distortion, also given by the perturbation expansion, the *separatrix of the resonance then can be associated with the stability boundary of the exact mapping*. By making this identification, we can estimate the location of the latter reasonably accurately.

Figure 2 is a remarkable case. Its most dramatic feature is the very large $2/7$ resonance which produces a system of seven islands. Seventh “order” resonances (i.e., resonances with winding number seven) should not appear until fifth order in the perturbation expansion, while the island chain is certainly more than a fifth order effect. In fact it is due to an *interference* between the $1/3$ resonance, which appears at first order in the perturbation expansion, and the $1/4$ resonance, which appears at second order. This is confirmed in Figure 2b which shows the perturbation theoretic prediction when those two resonances are explicitly taken into account.

Carrying out similar comparisons at other values of the

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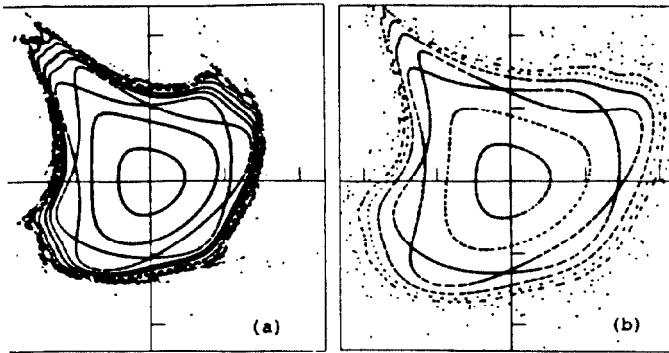


Figure 2: Same as Figure 1, but with $\nu = 0.29$.

tune we have found that separatrices associated with first and second order resonant normal form Hamiltonians can *usually* predict the stability boundary surprisingly accurately. Further studies were carried out on Hamiltonians including octupole kicks with similar results.[1] The large amplitude chaotic layers marking dynamic aperture could frequently be associated with separatrices from low order resonances. The conclusion drawn was that resonance seeding in two-dimensional systems worked better than we had a right to expect.

3 Four-dimensional maps

In order to control the beam-beam tune shift (and spread) and allow Fermilab to increase luminosity in the Tevatron, electrostatic separators will be used to place the fiducial (design) p and \bar{p} orbits on the branches of a double helix. Head-on collisions will occur only at two interaction regions, B0 and D0, where the helix will be pinched. Theoretical and experimental studies were conducted to study orbit stability under the double-helix scenario. The problem represents an interesting physical situation: even when individual kicks are small the combined beam-beam tune shift can be extremely large. Thus, the effects are distributed, much like those of small nonlinear fields distributed around the ring. In this type of situation, we can expect the dominant sources of instability to be driven by resonances. The principal observations we report here involve (a) bifurcations of chaotic separatrices leading to (b) the existence of very low entropy chaotic orbits. The control parameter for these bifurcations was the separator strength, or the size of the double helix. What follows is necessarily a cursory discussion of observations; a more detailed paper is being written.[4]

The non-trivial task of following chaotic, multi-dimensional separatrices through bifurcations was accomplished using AESOP, an "Exploratory Orbit Analysis" graphics shell for studying four-dimensional phase space. (See *C++ Objects for Beam Physics*, this Proceedings.) The low entropy orbits referred to above, previously called "tangled," were observed a few years ago using

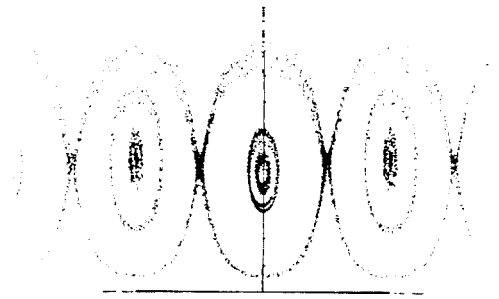


Figure 3: A slice through the separatrix for closed helix.

AESOP.[3, 5] What we were not able to do at that time was trace their evolution. Since then, a four-dimensional cursor was introduced into AESOP, making it easier to find and to follow separatrices.

At zero separation, we see a slice through the separatrix which exhibits the familiar four-lobe structure associated with a $2\nu_1 - 2\nu_2$ resonance, one excited strongly by the beam-beam interaction. (See Figure 3.) The coordinates for this three dimensional projection are the horizontal and vertical "angle" variables and an "action" variable. In AESOP's standard mode, the action variable displayed is simply proportional to the horizontal emittance, not a true action variable with nonlinearities present. This is basically the "angle-angle-action" plot which has been used before by the author, by Ruth *et al* [6], and by others. Figure 3 shows an orthographic projection of phase space along the $\delta_1 = \delta_2$ diagonal, with a number of orbits plotted. Regular, stable resonant orbits lie in manifolds which are near the regions $S_1 : \delta_1 - \delta_2 \simeq 0$ or $S_2 : \delta_1 - \delta_2 \simeq \pi$, while the unstable resonant orbits are near $U_1 : \delta_1 - \delta_2 \simeq \pi/2$ or $U_2 : \delta_1 - \delta_2 \simeq 3\pi/2$. Some care must be taken in interpretation. The apparently two stable regions just left and right of center are actually the *same* region, and the ones at the extreme ends coincide with the one in the center. We are viewing a projection of angle data which gets wrapped with 2π periodicity. The "four" lobes of this picture actually represent only two stable regions.

Seen from this projection, the orbits in regions S_1 and S_2 appear similar. In fact, they are very different. Tori surrounding S_1 are banded, as can be seen in the slightly rotated view of Figure 4. This indicates the existence of more complicated, "secondary" resonances winding around the "principal" tori, which envelop the "principal" resonant orbit. In addition, if one observes closely, these bands are themselves striated, pointing to a yet more complicated resonance structure. In contrast, the resonant orbit in region S_2 is actually a family of period eight orbits strung together. The small tori appearing in the side view of Figure 4 surround one set of orbits from this family.

When the electrostatic separators are powered and the helix begins to open, the first global bifurcation takes place, as illustrated in Figure 5. The separatrix no longer

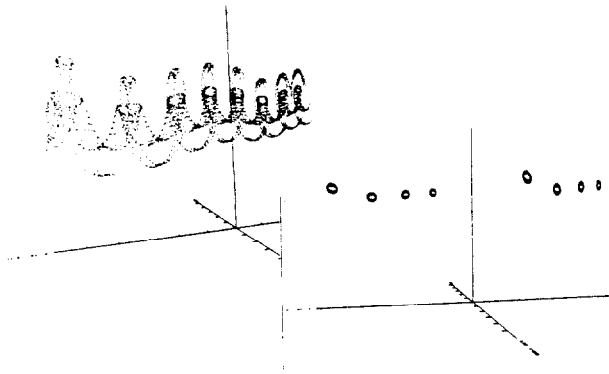


Figure 4: Side view of representative orbits from stable regions S_1 and S_2 .

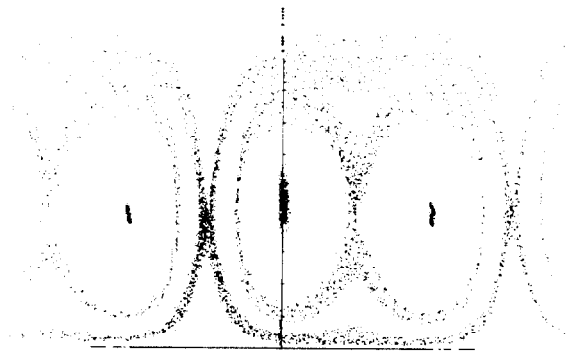


Figure 5: Appearance of the separatrix for small helix.

connects the two unstable resonant orbits. Rather, it has split into two branches, each of which is attached to one of them: the central “rotated figure eight” and its “cocoon.”¹ As the separation increases, a second bifurcation occurs in the vicinity of the unstable resonant orbit. (See Figure 6) Rather than forming the “cocoon” of Figure 5 the branch closes quickly, forming a new figure eight. Unlike the previous bifurcation, this is not one which takes a heteroclinic branch and changes it into a homoclinic one. It is homoclinic both before and after the bifurcation; what has changed is how it closes back on itself at large amplitudes.

Even though the separatrix is chaotic, its shape and behavior under bifurcation is clearly determined by an underlying, low order, integrably resonant system, supporting the reality of resonance seeding in four dimensions. (I suspect, but am not certain, that this is the first attempt to follow in detail the bifurcations of four-dimensional separatrices.)

The “tangled orbits” mentioned above (see the references for figures; there is no more space available here) appeared as the helix continued to open. They are almost periodic, with period five: watching one develop on the screen reveals that it returns on itself almost exactly af-



Figure 6: Appearance of the separatrix for larger helix.

ter five iterates. Correspondingly, the lengthening of the “thread” is slow; hundreds of iterates may be required to form a single “loop.” From the standpoint of the theory of chaos, we must interpret such orbits as chaotic with exceedingly small KS entropy. That is, the tangled orbits are *almost but not quite* regular. They appear, at least superficially, as one-dimensional objects. A great many iterates would be needed before the fractional parts could be determined with any accuracy.

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¹The situation may be even more complicated. There is some suggestion that one of these is actually two resonant orbits very close together, but this was not resolved.