DYNAMIC APERTURE EFFECTS DUE TO LINEAR COUPLING*

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I. INTRODUCTION

The coupling introduced by the random a_1 can produce considerable distortion of the betatron motion. For a given initial x_o, x'_o, y_o, y'_o, the maximum x and the maximum y for the subsequent motion can be considerably larger when coupling is present. The maximum x and y can be used as a measure of the betatron distortion. One effect of this betatron distortion shows up in the dynamic aperture, in a loss in the stability limit, A_{SL}, found by tracking. The betatron distortion causes the particle to move further out in the magnets, where it sees stronger non-linear fields. Previous tracking[1,2] showed a loss in A_{SL} due to random a_1 and b_1. It was noticed then that the loss in A_{SL} was associated with a betatron motion distortion which was primarily a linear effect. For a given initial x, x', y, y' the x_{max} and y_{max} in the high-β magnets were considerably larger (about 30% larger) for those random a_1 distributions which produced the smaller A_{SL}. The studies described below show that the stability limit, A_{SL}, depends on the starting location around the ring. This variation in A_{SL} around the ring can be correlated with the variation in the betatron distortion for particles starting at different places around the ring. It is proposed that the average of the A_{SL} found by starting at each of the QF, the focusing quadrupoles in the arcs, can be taken as a measure of the dynamic aperture. It is found that the average A_{SL} is reduced by about 15% by the random a_1 multipoles expected in RHIC.

Computing the stability limit A_{SL} is made more difficult by the dependence of A_{SL} on the starting location around the ring. In order to avoid having to compute A_{SL} for all possible starting locations at a QF, one can make use of an observed correlation between A_{SL} and a linear parameter, CDF, which is called the coupling distortion function, and which is defined below. Roughly, the CDF is a measure of how large x or y may become because of the presence of coupling for a given set of starting conditions x_o, x'_o, y_o, y'_o. The CDF being a linear parameter can be computed quickly starting at each QF in the ring. The QF with the largest CDF is found to have the smallest A_{SL}, and the smallest CDF corresponds to the largest A_{SL}. After finding the QF with the maximum and minimum CDF, one can compute the A_{SL} for these two starting locations with tracking runs. It is proposed that the average A_{SL} around the ring be approximated by taking the average of these two values of A_{SL} found at the locations where the CDF has its maximum and minimum.

II. CORRELATION OF CDF AND A_{SL}

The coupling distortion function, CDF, is defined by

$$\mathrm{CDF} = \frac{x_{max}(s)}{x_{max,nc}(s)} \tag{1}$$

for a given initial x, x', y, y' starting at $s = s_o$, and in the absence of non-linear fields. $x_{max,nc}$ is the x_{max} in the absence of coupling. A similar CDF exists also for the y-motion. The CDF depends on s and also on the starting conditions of x, x', y, y'. Since we are interested in the correlation between the CDF and the A_{SL} found in tracking studies, we will compute the CDF for the starting conditions $x' = y' = 0$, $\epsilon_x = \epsilon_y$, and at the s which corresponds to the location of the high-β quadrupoles in the insertions.

To compute CDF, one has to compute $x_{max}(s)$ in the presence of coupling. An analytical result was found for $x_{max}(s)$ for a given starting x, x', y, y'. This result is given in Section III.

Table 1: CDF at High β Quads for a particle starting at the focusing quadrupole at the middle of the first arc in RHIC for the RHIC $\beta^* = 2$ Lattice.

Field Error No.	CDF	β_1 (m)	β_2 (m)	A_{SL} (mm)
1	1.32	695	684	8.5
2	1.46	650	644	6.5
3	1.30	682	744	8.5
4	1.55	723	797	7.5
5	1.10	766	752	8.5
6	1.38	625	654	8.5
7	2.10	956	970	4.5
8	1.63	816	764	6.5
9	1.30	687	655	8.5
10	1.35	797	670	7.5

Table 1 lists the coupling distortion function, CDF, and the β-functions of the normal modes, β_1 and β_2 for ten different distributions of the random field errors expected in RHIC.[3] The CDF is the largest CDF found at any of the high β quadrupoles in the insertions for

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Table 2: The coupling distortion function, CDF, starting at different QF for a RHIC lattice with $\beta^* = 6$ and for a_1 errors corresponding to Seed 8.

QF Element No.	CDF	QF Element No.	CDF
1	1.86	113	1.67
15	1.48	127	1.67
29	1.81	141	1.62
43	1.52	145	1.67
57	1.81	169	1.62
71	1.57	1299	1.28
85	1.71	1313	2.00
99	1.62		

Table 3: The coupling distortion function, CDF, and the A_{SL} found starting at different Q for a RHIC lattice with $\beta^* = 6$ and Seed 8 a_1 errors.

QF Element No.	CDF	A_{SL} (mm)
1313	2.00	10.5
1	1.86	11.5
85	1.71	12.5
15	1.48	13.5
1299	1.28	14.5

either x or y motion for a particle starting at the focusing quadrupole in the middle of the first arc in RHIC. The β_1, β_2 are the largest β-functions of the normal modes found around the accelerator. For $a_1 = b_1 = 0$, the largest β functions are $\beta = 220$ m for $\beta^* = 6$ and $\beta^* = 630$ for $\beta^* = 2$. A CDF as large as CDF = 2.1 is found for one field error distribution. A simple picture of coupling would suggest CDF $\simeq$ 1.4 resulting from a complete exchange of the emittance between the x and y motions. The betatron distortion is about 40% larger for the worst case than what one would expect from the simple picture of a complete exchange of the emittances.

There is some correlation between CDF and β_1 and β_2. However, the CDF depends not only on β_1, β_2 but also on the mismatch in the emittances. For the given initial ϵ_x, ϵ_y, the corresponding ϵ_1, ϵ_2 may be larger than ϵ_x, ϵ_y where ϵ_1, ϵ_2 are the emittances for the normal modes.

Table 1 also shows the correlation between the CDF and the dynamic aperture which is measured by A_{SL}. A_{SL} is found from tracking studies with the random field errors, including higher multipoles, present. A_{SL} is the largest stable initial x with the starting conditions $\epsilon_x = \epsilon_y$, $x' = y' = 0$, and starting at the focusing quadrupole at the middle of the first arc in RHIC.

The correlation between the coupling distortion function, CDF, and the dynamic aperture A_{SL} is good. Note that the A_{SL} are computed after the coupling has been corrected[3] using the a_1 correctors in the insertions.

In the absence of the random a_1, b_1, the stability limit has been found to be $A_{SL} = 7.5$ mm for $\beta^* = 2$. The results in Table 1 indicate that for this particular starting place around the ring, A_{SL} has been reduced to $A_{SL} = 4.5$ mm for $\beta^* = 2$ m.

The above result for A_{SL} is misleading. It was found at a QF where the coupling distortion function, CDF, has a large value. If the tracking study to find A_{SL} is done starting on a different QF, a different value of A_{SL} will be found which is inversely correlated with the CDF at that QF.

If the particle is started at different QF, the value of CDF and A_{SL} will change at each QF. Table 2 shows this variation in CDF at 14 different QF. The CDF listed is the largest CDF found at high-β quadrupoles in the insertions, when the particle is started at different QF.

A_{SL} will also vary around the ring, since A_{SL} and CDF are correlated. In Table 3 A_{SL} and CDF are listed for a particle starting at 5 different QF in the ring. These 5 points include the largest and smallest CDF found in the ring. A_{SL} is computed at the QF at which the particle is started.

Since A_{SL} now depends on which QF the particle is started at, one has the problem of which value of A_{SL} is to be used for the dynamic aperture. The procedure proposed here is to use the average A_{SL} found from the A_{SL} computed at all the QF in the ring.

To compute $A_{SL,av}$ the average A_{SL}, one should in principle compute A_{SL} at all the QF in the ring and then compute the average A_{SL}. This would require a good deal of effort. Instead, the following procedure was used. The coupling distortion function at the high β quadrupole is computed starting at each QF in the ring. One then locates the QF that give the largest and smallest CDF, CDF_{max} and CDF_{min}. The stability limit A_{SL} is computed starting at these two QF. It is assumed that this gives the largest and smallest A_{SL}, $A_{SL,max}$ and $A_{SL,min}$. The average A_{SL} is then computed as

$$A_{SL,av} = \frac{1}{2}(A_{SL,max} + A_{SL,min})$$

The computation of $A_{SL,av}$ for the two worst field errors, errors 7 and 8, are shown in Table 4 for two RHIC lattices having $\beta^* = 6$ and $\beta^* = 2$.

The loss in A_{SL} is now 15%. A_{SL} has been decreased from 15.5 mm to 13 mm, for $\beta^* = 6$ from 7.5 mm to 6.5 mm for $\beta^* = 2$.

Table 4: Computation of $A_{SL,av}$.

β^*	Error No.	A_{SL} max	A_{SL} min	A_{SL} av	CDF Max	CDF Min
6	7	14.5	11.5	13	1.67	1.29
6	8	16.5	10.5	13.5	2.0	1.24
2	7	8.5	4.5	6.5	2.1	1.10
2	8	8.5	6.5	7	1.8	1.21

III. ANALYTICAL RESULTS FOR X_{max} and Y_{max}

In this section, analytical results are given for the maximum x and y, x_{max} and y_{max}, that will be reached for a particle with a given initial x, x', y, y' in the presence of coupling. These results are needed in order to compute the coupling distortion function, CDF. Derivations of these results will be given in a future paper.

Edwards and Teng showed how to transfer to a new set of coordinates v, v', u, u' which are uncoupled.[5] These new normal coordinates are related to x, x', y, y' by a 4×4 matrix R

$$x = R\, v \ .$$

The normal coordinates have emittances ϵ_1 and ϵ_2 and β-functions β_1 and β_2. ϵ_1 and ϵ_2 are invariants. The x and y motion can be written as the sum of these two normal modes.

β_1 and β_2 and the R matrix can be computed from the one turn transfer matrix.[5] For RHIC, β_1 and β_2 can be considerably larger than β_x, β_y by as much as 100% in the worse case found. However, the x_{max}, y_{max} for a given initial x, x', y, y' are not simply related to β_1, β_2.

For a given set of initial x, x', y, y', x_{max} and y_{max} are given by

$$\begin{aligned} x_{max} &= (\beta_1\epsilon_1)^{\frac{1}{2}} \cos\phi + (\beta_{x,2}\epsilon_x)^{\frac{1}{2}} \sin\phi, \\ \beta_{x,2} &= \overline{D}_{11}^2\beta_2 + \overline{D}_{12}^2\gamma_2 + 2\overline{D}_{11}\overline{D}_{12}\alpha_2, \\ y_{max} &= (\beta_2\epsilon_2)^{\frac{1}{2}} \cos\phi + (\beta_{y,1}\epsilon_1)^{\frac{1}{2}} \sin\phi, \\ \beta_{y,1} &= D_{11}^2\beta_1 + D_{12}^2\gamma_1 + 2D_{11}D_{12}\alpha_1 \ . \end{aligned} \tag{2}$$

D and $\overline{D}$ are 2×2 matrices, which together with ϕ define the 4×4 R matrix R is given by

$$R = \begin{pmatrix} I \cos\phi & D \sin\phi \\ -\overline{D} \sin\phi & I \cos\phi \end{pmatrix} \tag{3}$$

$\overline{D} = D^{-1}$, $|D| = 1$ and I is the 2×2 identity matrix.

β_1, α_1, γ_1 and β_2, α_2, γ_2 are the orbit parameters of the normal modes, ϵ_1 and ϵ_2 are the emittances of the normal modes that correspond to the initial x, x', y, y'.

One can also find expressions for x'_{max} and y'_{max}. These are given by

$$\begin{aligned} x'_{max} &= (\gamma_1\epsilon_1)^{\frac{1}{2}} \cos\varphi + (\gamma_{x,2}\epsilon_2)^{\frac{1}{2}} \sin\phi, \\ \gamma_{x,2} &= \overline{D}_{21}^2\beta_2 + \overline{D}_{22}^2\gamma_2 + 2\overline{D}_{21}\overline{D}_{22}\alpha_2, \\ y'_{max} &= (\gamma_2\epsilon_2)^{\frac{1}{2}} \cos\varphi + (\gamma_{y,1}\epsilon_1)^{\frac{1}{2}} \sin\phi, \\ \gamma_{y,1} &= D_{21}^2\beta_1 + D_{22}^2\gamma_1 + 2D_{21}D_{11}\alpha_1 \ . \end{aligned} \tag{4}$$

Edwards and Teng[5] describe how to compute the R matrix and β_1, α_1, γ_1 and β_2, α_2, γ_2 from the one turn transfer matrix. This then allows one to compute ϵ_1 and ϵ_2 from the initial x, x', y, y'. Equation (2) can then be used to compute x_{max} and y_{max} for the given initial x, x', y, y', which is needed to compute the coupling distortion function, CDF (see Eq. (1)).

IV. REFERENCES

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