BETA FUNCTIONS IN THE PRESENCE OF LINEAR COUPLING*

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I. INTRODUCTION

One effect of random skew quadrupole field errors is to perturb the β-functions. This effect may be large in proton accelerators using superconducting magnets, because of the relatively large random skew quadrupole field errors that are expected in these magnets. The effect is also increased by the required insertions in proton colliders which generate large β-functions in the insertion region.

This effect has been studied in the RHIC accelerator (the Relativistic Heavy Ion Collider proposed at Brookhaven National Laboratory). For RHIC, large changes in the β-functions were found, as large as 100% increase in the β-function in one worse case.

An analytic result has been found for the changes in the beta functions caused by the random a_1. This result indicates that the important harmonics of a_1 that need to be controlled are the harmonics near $\nu_x + \nu_y$. This has been verified in computer studies. The random a_1 will also generate a large tune splitting[1,2] in RHIC. The analytic result indicates that the correction of the tune splitting will also correct a large part of the errors in the beta functions. This has been shown in computer studies. The a_1 correction system that has been developed to correct the tune splitting in RHIC appears able to correct most of the error in the beta functions.

II. RESULTS FOR THE β-FUNCTIONS

The random quadrupole field errors expected in RHIC[2] are used in this study. The effect of the random a_1, the skew quadrupole error, is to couple the x and y motions. The x motion and the y motion can each be written as the sum of 2 normal modes which have the ν-values ν_1 and ν_2. Each of the normal modes also have β-functions denoted by β_1 and β_2. For a given distribution of field errors, β_1 and β_2 can be computed using the results of Edwards and Teng.[3]

An analytic result has been found for the change in β_1 and β_2 which is valid close to the resonance $\nu_x - \nu_y = p$, p being some integer. Let β_1 be the beta function that approaches β_x, the unperturbed beta function, when $a_1 \to 0$, and similarly for β_2 and β_y. Then $(\beta_1 - \beta_x)/\beta_x$ and $(\beta_2 - \beta_y)/\beta_y$ are given by

$$(\beta_1 - \beta_x)/\beta_x = \sum_n \left[\left(\frac{\nu_1 - \nu_x}{\Delta\nu(\nu_1, \nu_1 - p)} \right) b_n \frac{\exp[i(n-p)\theta_x]}{n + \nu_x + \nu_y} + c.c. \right]$$

$$(\beta_2 - \beta_y)/\beta_y = \sum_n \left[\left(\frac{\nu_2 - \nu_y}{\Delta\nu(\nu_2 + p, \nu_2)} \right) c_n \frac{\exp[i(n+p)\theta_y]}{n + \nu_x + \nu_y} + c.c. \right]$$

$$b_n = (1/4\pi\rho) \int ds\ a_1 (\beta_x\beta_y)^{\frac{1}{2}} \exp[i(-(n+\nu_y)\theta_x + \nu_y\theta_y)]$$

$$c_n = (1/4\pi\rho) \int ds\ a_1 (\beta_x\beta_y)^{\frac{1}{2}} \exp[i(-(n+\nu_x)\theta_y + \nu_x\theta_x)]$$

$$\Delta\nu(\nu_x, \nu_y) = (1/4\pi\rho) \int ds\ a_1 (\beta_x\beta_y)^{\frac{1}{2}} \exp[i(-\nu_x\theta_x + \nu_y\theta_y)]$$

$$\theta_x = \psi_x/\nu_x, \theta_y = \psi_y/\nu_y \tag{1}$$

ν_x and ν_y are the unperturbed tunes. The above results will be derived in a future paper. Eqs. (1) can be written in integral form as

$$(\beta_1 - \beta_x)/\beta_x = -\frac{\nu_1 - \nu_x}{|\Delta\nu(\nu_1, \nu_1 - p)|} \frac{1}{2\rho \sin\pi(\nu_x + \nu_y)} \times \int ds' a_1(s') \left(\beta_x(s')\,\beta_y(s')\right)^{\frac{1}{2}} \cos\left[\pm\pi(\nu_x+\nu_y) - (\nu_x+\nu_y)(\theta - \theta') + \nu_y(\theta'_y - \theta'_x) - \delta_1\right] \tag{2}$$

The integral form for $(\beta_2 - \beta_y)/\beta_y$ is obtained from Eq. (2) through the substitutions in Eq. (2) of $\beta_1 \to \beta_2$, $\beta_x \to \beta_y$, $\nu_x \to \nu_y$, $\nu_y \to \nu_x$, $\delta_1 \to \delta_2$, $\Delta\nu(\nu_1, \nu_1 - p) \to \Delta\nu(\nu_2 + p, \nu_2)$, $\nu_1 \to \nu_2$, $\delta_1 =$ phase $[\Delta\nu(\nu_1, \nu_1 - p)]$ and $\delta_2 =$ phase $[\Delta\nu^*(\nu_2 + p, \nu_2)]$. For the $\pm$, the top sign is used for $\theta > \theta'$ and the bottom sign for $\theta < \theta'$.

Eq. (1) shows that the important harmonics in a_1 are the harmonics near $\nu_x + \nu_y$, and a correction system for β_1, β_2 should probably control the harmonics of a_1 near $\nu_x + \nu_y$. However, Eq. (1) shows that the dominant harmonic in $(\beta_1 - \beta_x)/\beta_x$ is the $2\nu_x$ harmonic and in $(\beta_2 - \beta_y)/\beta_y$ the $2\nu_y$ harmonic. Near the $\nu_x - \nu_y = 0$ resonance, the $2\nu_x$, $2\nu_y$ and $\nu_x + \nu_y$ harmonics are about the same. However they are different near the $\nu_x - \nu_y = p$ resonance. The above formulae appear to suggest the effects of a $\nu_x + \nu_y =$ integer sum resonance. However it may be more proper to label it as a half-integer resonance effect in the coordinate system of the normal modes.

One may note the factor $(\nu_1 - \nu_2)/\Delta\nu$. For large $\Delta\nu$, $\Delta\nu >> (\nu_x - \nu_y - p)$, then this function approaches 1. This may be seen from the result for ν_1 and ν_2 near

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the $\nu_x - \nu_y = 0$ resonance.

$$\begin{aligned} \nu_1 &= \overline{\nu} \pm \left[(\nu_x - \nu_y)^2 /4 + |\Delta\nu|^2 \right]^{\frac{1}{2}} , \\ \nu_2 &= \overline{\nu} \mp \left[(\nu_x - \nu_y)^2 /4 + |\Delta\nu|^2 \right]^{\frac{1}{2}} , \\ \overline{\nu} &= (\nu_x + \nu_y)/2, \quad \Delta\nu = \Delta\nu(\overline{\nu},\overline{\nu}) \ . \end{aligned} \tag{3}$$

For the $\pm$, the top sign is used for $\nu_x > \nu_y$ and the bottom sign for $\nu_x < \nu_y$. According to Eq. (3) $(\nu_1 - \nu_x)/\Delta\nu \to 1$ for large $\Delta\nu$ and $(\nu_1 - \nu_x)/\Delta\nu \simeq 2\Delta\nu/|(\nu_x - \nu_y)|$ for small $\Delta\nu$. For very small $\Delta\nu$, the result is not easy to compute correctly because of higher order term in a_1 that are not given in Eqs. (3). Computer studies show that correction of ν_1 and ν_2 to make them closer to ν_x, ν_y tends to decrease $(\beta_1 - \beta_x)/\beta_x$ and $(\beta_2 - \beta_y)/\beta_y$ which may be due in part to this $(\nu_1 - \nu_x)/\Delta\nu$ factor.

Table 1: Maximum β_1, β_2 for 10 distributions of random a_1 fields for the RHIC $\beta^* = 2$ lattice. No corrections are present.

Error Field	$\beta_{1,max}$ at QF (m)	$\beta_{2,max}$ at QD (m)
1	68	75
2	138	95
3	89	78
4	83	74
5	69	65
6	65	67
7	76	82
8	78	79
9	80	78
10	77	87

One may also note that $(\beta_1 - \beta_x)/\beta_x$ is linear in a_1 for large $\Delta\nu$, and quadratic in a_1 for small $\Delta\nu$.

A result for the rms value of $(\beta_1 - \beta_x)/\beta_x$ due to random distribution of a_1 errors may be obtained from the integral form Eq. (2), for the case when $|\Delta\nu| >> |\nu_x - \nu_y - p|$. In this case $|\nu_1 - \nu_x|/|\Delta\nu| \simeq 1$ and

$$\begin{aligned} \left(\frac{\beta_1 - \beta_x}{\beta_x}\right)^2_{rms} &= \sum_k \left(\frac{\beta_1 - \beta_x}{\beta_x}\right)^2_{k,rms} \\ \left(\frac{\beta_1 - \beta_x}{\beta_x}\right)_{k,rms} &= N_k^{1/2} \frac{\left((\beta_x\beta_y)^{1/2} a_{1,rms}\right)_k}{2.8\rho \sin\pi(\nu_x + \nu_y)} \end{aligned} \tag{4}$$

where the index k indicates the different types of magnets. N_k is the number of magnets of a certain type. Eq. (4) also gives the result for $((\beta_2 - \beta_y)/\beta_y)_{rms}$. One also sees that

$$((\beta_1 - \beta_x)/\beta_x)_{rms} = [4\pi/(2.8 \sin\pi(\nu_x + \nu_y))] \Delta\nu_{rms} \tag{5}$$

where $\Delta\nu_{rms}$ is the rms value of $\Delta\nu$.

Table 2: Maximum β_1 at QF and the maximum β_2 at QD for 10 distributions of random a_1 fields, for the RHIC $\beta^* = 2$ lattice, showing the effect of correcting the tune splitting on β_1, β_2.

Error Field	2 Family Correction System		Enlarged a_1 Correction System	
	$\beta_{1,max}$	$\beta_{2,max}$	$\beta_{1,max}$	$\beta_{2,max}$
1	58	58	57	56
2	57	54	59	74
3	56	59	56	59
4	63	57	61	56
5	63	60	63	60
6	55	58	58	62
7	102	84	61	59
8	60	57	64	60
9	58	59	58	58
10	60	78	57	56

Table 1 lists the beta functions, before any correction of the random a_1 fields for the RHIC $\beta^* = 2$ lattice for ten distributions of the random a_1. The table lists the maximum β_1 found at the normal focusing quadrupoles, QF, and the maximum β_2 at the normal defocusing quadrupoles, QD. The unperturbed value of the maximum β_x and β_y is 50 m. One sees that change in the beta functions of the order of 100% are computed. β_1 and β_2 were computed using the exact results of Edwards and Teng.[3]

Using Eq. (4), one can compute the expected rms value of $(\beta_1 - \beta_x)/\beta_x$ which gives for RHIC $((\beta_1 - \beta_x)/\beta_x)_{rms} = 0.29$. For 72 QF magnets, the 90% probability result is $\beta_{1,max} = \beta_{2,max} = 95$. This is in fairly good agreement with the results in Table 1. The exceptionally large $\beta_{1,max} = 138$ for seed 2 is due to ν_1 being shifted close to the $\nu_1 = 29$ resonance where β_1 would become infinite. This effect of the $\nu_1 = 29$ resonance is not included in our results for $(\beta_1 - \beta_x)/\beta_x$.

III. CORRECTION OF β_1, β_2

Computer studies indicate that when the tune shifts $\nu_1 - \nu_x$ and $\nu_2 - \nu_y$ are corrected, then $\beta_1 - \beta_x$ and $\beta_2 - \beta_y$ are also reduced. This is indicated by the analytical result Eqs. (1). It is partly due to the $(\nu_1 - \nu_x)/\Delta\nu$, $(\nu_2 - \nu_y)/\Delta\nu$ factors in Eqs. (1). It is also partly due to the driving terms b_n, c_n in Eq. (1), which are also important driving terms for the tune splitting.[4] In RHIC the tune splitting correction includes a 2 family a_1 correction system, that corrects the different resonance, $\nu_x - \nu_y = 0$, and most of the tune splitting, and an enlarged a_1 correction system[4] to correct the residual tune splitting by controlling the harmonics of a_1 near $\nu_x + \nu_y$.

Table 2 shows what happens to β_1, β_2 when the tune splitting is corrected by the above 2 family tune splitting correction systems. One sees that the $\beta_1 - \beta_x$, $\beta_2 - \beta_y$ errors are considerably reduced. The results in Table 2, include the effects of random b_1 field which can generate a 20% error in β_1 and β_2. However, a considerable error in β_1, β_2 remains for some distributions.

Table 2 also shows what happens to β_1, β_2 when the tune splitting is further reduced using the enlarged a_1 correction system to control harmonics near $\nu_x + \nu_y$. One sees a considerable improvement in β_1 and β_2.

The results for β_1, β_2 can be further improved by adjusting the harmonics of a_1 near $\nu_x + \nu_y$, or by moving the ν–values slightly closer to the resonance line $\nu_x - \nu_y = 0$. At this level, the correction of the tune splitting and the β_1, β_2 correction may be competing with each other. Using these procedures the largest error in β_1, β_2 was reduced to 28%, which includes the 20% effect due to the random b_1 which has not been corrected.

IV. REFERENCES

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