Microwave Instability at Transition – Stability Diagram Approach

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Abstract

A simple model of a beam at transition driven by a storage ring impedance is formulated in the framework of the nonlinear Vlasov equation. This yields a set of coupled equations of motion describing time evolution of a single coherent mode and the overall equilibrium density distribution function. At transition, contour integration in the dispersion relation can be carried out analytically and a simple closed formula for the coherent frequency is obtained. From the resulting stability diagram further conclusions about the growth time of the microwave instability and the longitudinal emittance blowup at transition are derived.

I. INTRODUCTION

One would like to sort out purely kinematic contributions to the emittance blowup (due to the Johnsen and Umstätter effects)[1] from the intensity dependent one caused by the microwave instability, possibly building up at transition. The longitudinal phase space has a very peculiar structure; at transition the RF bucket does not exist in the usual sense and a beam can be considered a coasting beam to a good approximation. When the influence of the external restoring force disappears the beam is very susceptible to any fast growing instability which may in turn (through nonlinear driving terms) reshape the overall longitudinal phase space of the beam.

II. LONGITUDINAL BEAM DYNAMICS AT TRANSITION

Consider an initially uniform distribution of particles inside a storage ring modeled by a statistical density distribution function defined in a classical phase space as

$$f(\varepsilon,\theta,t) = f^{o}(\varepsilon,t) + \sum_{n\neq 0} h_n(\varepsilon,t)\, e^{i\theta n} \quad , \tag{1}$$

where θ is the azimuthal angle around the ring circumference and ε represents the energy deviation from its synchronous value, E_o. Here $f(\varepsilon,\theta,t)$ is normalized to the total number of particles in the storage ring, N. The phase space continuity equation which governs $f(\varepsilon,\theta,t)$ can be written as follows

$$\frac{\partial}{\partial t} f(\varepsilon,\theta,t) + \omega \frac{\partial}{\partial \theta} f(\varepsilon,\theta,t) + \dot{\varepsilon}\, \frac{\partial}{\partial \varepsilon} f(\varepsilon,\theta,t) = 0 \ . \tag{2}$$

The revolution frequency, ω, of a given particle depends on its momentum offset, Δp, via the momentum compaction factor, α. The fractional frequency shift is given by

$$\frac{\Delta \omega}{\omega_o} = -\left(\alpha - \frac{1}{\gamma^2}\right)\delta \ ,$$

where (3)

$$\delta = \frac{\Delta p}{p_o} = \frac{1}{\beta^2}\frac{\varepsilon}{E_o} \ .$$

In principle the momentum compaction factor also exhibits some chromatic dispersion according to general expansion

$$\alpha = \frac{p}{C}\frac{dC}{dp} = \alpha_o + \alpha_p\, \delta + O(\delta^2) \ . \tag{4}$$

Right at transition, $\gamma = \gamma_t$, the linear term in Eq.(3) disappears, since

$$\alpha_o - \frac{1}{\gamma^2} = 0 \tag{5}$$

and the leading term in $\omega(\varepsilon)$ happens to be quadratic in ε. One can summarize Eqs.(3)–(5) to the lowest leading term in ε as follows

$$\omega(\varepsilon) = \omega_o - K\,\varepsilon^2 \ , \tag{6}$$

where the coefficient K is given by the following expression

$$K = \frac{\alpha_p \omega_o}{\beta^4 {E_o}^2} \ . \tag{7}$$

Here α_p is a purely lattice dependent parameter, which is easy to express in terms of azimuthal averages of the lattice dispersion function[3].

The beam environment in Eq.(2) is modeled by the wakefield impedance of a storage ring represented in frequency domain by $Z(\omega)$. The energy of the beam changes by

$$\dot{\varepsilon} = -\, e\omega_o \sum_{n\neq 0} Z_n \phi_n(t)\, e^{i\theta n} \quad , \qquad Z_n = Z(n\omega_o). \tag{8}$$

where

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$$\phi_n(t) = -e\omega_o \frac{1}{2\pi} \int_{-\infty}^{\infty} d\varepsilon\, h_n(\varepsilon,t)\,. \qquad (9)$$

Substituting Eqs.(1) and (8) into Eq.(2) and using orthogonality of azimuthal plane waves, one can rewrite Vlasov equation as a set of coupled equations of motion for individual azimuthal harmonics of the distribution function.

$$\frac{\partial}{\partial t} f^o(\varepsilon,t) - e\omega_o \sum_{n\neq 0} Z_n{}^* \phi_n{}^*(t) \frac{\partial}{\partial \varepsilon} h_n(\varepsilon,t) = 0 \quad , \qquad (10)$$

and

$$\frac{\partial}{\partial t} h_n(\varepsilon,t) + in\omega\, h_n(\varepsilon,t) - e\omega_o\, Z_n\, \phi_n(t) \frac{\partial}{\partial \varepsilon} f^o(\varepsilon,t)$$

$$- e\omega_o \sum_{m\neq 0} Z_{n-m}\, \phi_{n-m}(t) \frac{\partial}{\partial \varepsilon} h_m(\varepsilon,t) = 0 \quad . \qquad (11)$$

Here we introduce the instantaneous coherent frequency, $\Omega_n(t)$, describing evolution of the n-th mode within a small time interval (t, t') according to the formula

$$h_n(\varepsilon,t') = e^{-i\,\Omega_n(t)(t-t')}\, h_n(\varepsilon,t), \qquad t \approx t'\,. \qquad (12)$$

We also require that $f^o(\varepsilon,t)$ is a slowly varying function of time compared to rapidly oscillating coherent modes. Including both assumptions one can rewrite Eq.(11) as follows

$$h_n(\varepsilon) = (e\omega_o)^2 \frac{\partial}{\partial \varepsilon} f^o(\varepsilon,t) \frac{Z_n}{n\omega - \Omega_n(t)} \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\varepsilon'\, h_n(\varepsilon')\,. \quad (13)$$

As was pointed out by Landau,[5] an appropriate integration of Eq.(13) leads to the following dispersion relationship defining the coherent frequency, $\Omega_n(t)$

$$1 = \left(\frac{e\omega_o}{2\pi}\right)^2 N\, Z_n \int_C d\varepsilon\, \frac{\frac{\partial}{\partial \varepsilon}\psi(\varepsilon,t)}{i[n\omega(\varepsilon) - \Omega_n]} \quad . \qquad (14)$$

Here, Ω_n and $\omega(\varepsilon)$, given explicitly by Eq.(6), define configuration of poles in the complex ε-plane.

III. STABILITY DIAGRAM

The dispersion relation, given by Eq.(14), can be solved with respect to the coherent frequency, Ω, then contours of constant growth rate, $\mathrm{Im}(\Omega)$, can be composed in complex impedance plane. Here we assume a simple Gaussian distribution parametrized by $\sigma = \langle\varepsilon\rangle_{rms}$ as follows

$$\psi(\varepsilon) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{\varepsilon^2}{2\sigma^2}\right). \qquad (15)$$

Assuming a frequency dispersion given by Eq.(6), one can rewrite the dispersion equation in the following form

$$1 = \left(\frac{e\omega_o}{2\pi}\right)^2 N\, Z_n \frac{1}{2\sigma^2 niK} \int_C d\varepsilon\, \frac{2\varepsilon\psi(\varepsilon,t)}{(\varepsilon + \Xi)(\varepsilon - \Xi)} \quad . \qquad (16)$$

where

$$\Xi^2 = \frac{n\omega_o - \Omega}{nK}\,.$$

As was shown in the Appendix the contour integral in Eq.(16) is given by Eq.(29) as follows

$$\int_C d\varepsilon\, \frac{2\varepsilon\psi(\varepsilon,t)}{(\varepsilon + \Xi)(\varepsilon - \Xi)} = 2\pi i\, \psi(\Xi,t)\,. \qquad (17)$$

Therefore, right at transition, the microwave stability is strictly governed by the Landau damping residuum term given by the right-hand side of Eq.(17). The last statement would translate to a global microwave stability at transition. Substituting Eq.(17) into Eq.(16) allows one to express the coherent frequency of a given mode in the following form

$$\Omega_n = n\omega_o - 2nK\sigma^2 \ln\left(\frac{(e\omega_o)^2}{(2\pi)^{3/2}} \frac{NZ_n}{2nK\sigma^3}\right). \qquad (18)$$

Taking into account multivaluedness of the above expression one has to introduce appropriate cut-lines (connecting different Riemann sheets of the general solution) at the branch points. The growth rate, introduced as the imaginary part of the coherent frequency, is given by the following expression

$$\frac{1}{\tau_n} = -2\alpha_p n\omega_o \left(\frac{\sigma}{E}\right)^2 F_n(\omega)$$

with (19)

$$F_n(\omega) = \arctan\left(\frac{Y_n}{X_n}\right), \quad Z_n = X_n + iY_n\,.$$

For the purpose of this calculation we assume a resonant impedance centered at the frequency ω_r.

One can see from Figure 1 that the general solution for the phase factor, $F_n(x)$, spans an infinite family of curves joined by the vertical cut-lines at $x = \pm 1$. To select a physical solution we impose a following condition; the $x = 0$ point has to be equivalent to the $x \to \pm\infty$ asymptotics and it must be included in the stable region (zero impedance). This narrows down the allowed solutions to the one highlighted in Figure 1. A whole frequency spectrum is contained in the stable region (upper half-plane) with the resonant points, $x = \pm 1$, touching the stability curve.

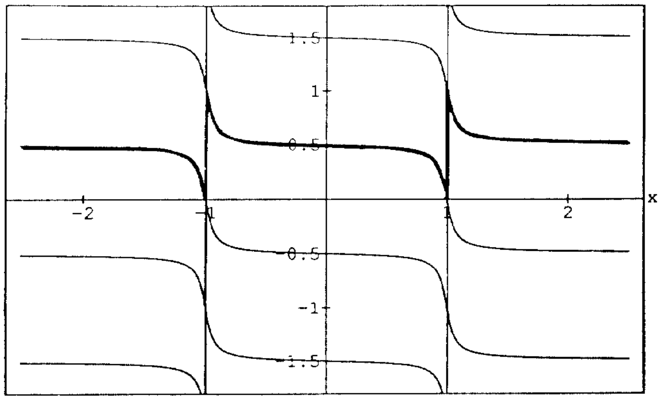

Figure 1 Microwave stability at transition, illustrated by a family of curves representing the phase factor, $F_n(\omega)$, in units of π, plotted versus $x = \omega/\omega_r$.

IV. CONCLUSIONS

We have shown that right at transition, the dispersion integral includes only the Landau damping term making a beam stable against the microwave instability. No instability develops, therefore the longitudinal emittance is not increased by the microwave instability.

V. APPENDIX

Here we evaluate the dispersion integral at transition, which is introduced as follows

$$I = \int_C d\varepsilon \, \frac{2\varepsilon\psi(\varepsilon)}{(\varepsilon + \Xi)(\varepsilon - \Xi)} \qquad (20)$$

where $\psi(\Xi,t)$ is a Gaussian parametrized by

$$\psi(\varepsilon) = \sqrt{\frac{\alpha}{\pi}}\, e^{-\alpha\varepsilon^2} . \qquad (21)$$

Using simple algebra one can rewrite our integral, I, in terms of the plasma dispersion integrals as follows

$$I = \sqrt{\frac{\alpha}{\pi}}\, [\, D(\sqrt{\alpha}\,\Xi) + D(-\sqrt{\alpha}\,\Xi)\,] . \qquad (22)$$

where the plasma dispersion integral is defined by the following formula

$$D(\xi) = \int_C d\varepsilon \, \frac{e^{-\alpha\varepsilon^2}}{(\varepsilon - \xi)} \qquad (23)$$

The contour C contains the real axis, an infinite semi-circle closed in the upper half-plane and a detour piece enclosing any singularity in the lower half-plane. Contributions along the first two pieces of the contour are given by the principle value integral, while the integral along the detour piece is equal to the residuum of the integrant at the singularity. Both contributions are summarized below

$$D(\xi) = W(\xi) \ + 2\pi i \theta_\xi\, e^{-\xi^2} \qquad (24)$$

where

$$W(\xi) = P \int_{-\infty}^{\infty} d\varepsilon \, \frac{e^{-\alpha\varepsilon^2}}{(\varepsilon - \xi)} \qquad (25)$$

and θ_ξ is defined as follows

$$\theta_\xi = \begin{cases} 0 & \text{if} \quad \mathrm{Im}(\xi) > 0 \\ \frac{1}{2} & \text{if} \quad \mathrm{Im}(\xi) = 0 \\ 1 & \text{if} \quad \mathrm{Im}(\xi) < 0 \end{cases} \qquad (26)$$

Substituting a sequence of above equations into Eq.(22) one gets the following expression

$$I = \sqrt{\frac{\alpha}{\pi}}\, [W(\sqrt{\alpha}\,\Xi) + W(-\sqrt{\alpha}\,\Xi) + 2\pi i\,(\theta_\xi + \theta_{-\xi})e^{-\alpha\Xi^2}] . (27)$$

One can easily check from the definitions, Eqs.(25) and (26), that $W(\xi)$ is an odd function of ξ and the following simple identity for θ_ξ holds

$$\theta_{-\xi} = 1 - \theta_\xi \qquad (A28)$$

Applying these identities to Eq.(27) reduces it to the following final expression

$$I = 2\pi i\ \psi(\Xi). \qquad (29)$$

VI. REFERENCES

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