Electromagnetic Instability of an Intense Beam in a Quadrupole Focusing System

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Abstract

Discrete quadrupole focusing systems are subject to the electromagnetic instability previously found in helical quadrupole strong focusing systems. The dispersion relation is derived. Analytical growth rates are obtained. Instability boundaries are established. Numerical solutions of the dispersion relation results show that the overall stability properties are not favorable for long pulse, high-current electron beams.

I. Dispersion Relation

Discrete quadrupole focusing systems, also called FODO lattices, have been used to transport charged particle beams for a variety of applications. Helical quadrupole (stellarator) focusing systems are subject to an electromagnetic instability, which we referred to as the three-wave instability.[1-3] This has been observed experimentally.[4] Here, we consider the corresponding instability[5] for transverse perturbations of a beam centroid interacting with a FODO lattice field and a TE_{11} waveguide mode.

We consider an alternating gradient quadrupole field (B_{qx}, B_{qy}), where

$$B_{qx} = -B_q k_q f(z) y, \qquad B_{qy} = -B_q k_q f(z) x, \qquad (1a-b)$$

$B_q k_q$ is the peak quadrupole field, $f(z)$ is periodic with maximum value of one, $k_q = 2\pi/\lambda_q$ and λ_q is the period of the quadrupole field. The representation for the quadrupole field in Eqs. (1a-b) is valid near the z-axis, i.e., $(x^2+y^2)^{1/2} << \lambda_q/2\pi$. In equilibrium, the electron beam travels along the axis of a circular waveguide at velocity v_o and is monoenergetic with $\gamma_o = (1-\beta_o^2)^{-1/2}$, where $\beta_o = v_o/c$. For simplicity, we let $f(z) = \cos(nk_q z)$.

We assume the electron beam propagates within a perfectly conducting cylindrical waveguide of radius r_g. Both the beam radius and beam centroid displacement are assumed to be small in comparison to the waveguide radius. We expect TE_{11} mode to have the largest growth rate, because its electric field peaks on axis.

The dispersion relation for the TE_{11} mode in the presence of the beam is a function of the determinant of a tri-diagonal matrix of the form

$$det\begin{pmatrix} \ddots & \ddots & 0 & 0 & \cdots \\ \ddots & T_{-2} & \frac{K_d^4}{2}S_{-3} & 0 & \cdots \\ \cdots & \frac{K_d^4}{2}S_{+1} & T_0 & \frac{K_d^4}{2}S_{-1} & \cdots \\ \cdots & 0 & \frac{K_d^4}{2}S_{+3} & T_{+2} & \ddots \\ \cdots & 0 & 0 & \ddots & \ddots \end{pmatrix} = 0, \qquad (2)$$

where

$$S_m(\omega,k) = S(\omega, k+mk_q), \qquad (3)$$

$$S(\omega,k) = -(\frac{\omega}{v_o}-k)^2 + k_b^2\frac{(\omega/v_o-k)^2}{(\omega^2/c^2-k^2-\mu_{11}^2)}, \qquad (4)$$

$$T_m(\omega,k) = S_m S_{m+1} S_{m-1} - \frac{1}{2}K_d^4\left[S_{m-1}+S_{m+1}\right], \qquad (5)$$

and $K_d^2 = K_q k_q/\sqrt{2}$. Here, ω and k are the frequency and wavenumber of the TE_{11} mode, $K_q = \Omega_q/v_o$ is the wavenumber associated with the quadrupole field, where $\Omega_q = |e|B_q/\gamma_o m_o c$, $k_b^2 = 2\nu/\gamma_o I_{11}$, $\nu \simeq I_e/17\beta_o$, I_e is the electron beam current in units of kilo-Amperes, $I_{11} = \mu_{11}^{-2}(\mu_{11}^2 r_g^2 - 1)J_1^2(\mu_{11}r_g)$, and $\mu_{11}r_g$ is the the largest positive zero of Bessel function J_1'.

It can be shown from Eq. (2) that the growth rate of the instability is periodic in k. For a given unstable frequency ω_o, the unstable wavenumbers are at all $k = k_0 + nk_q$, where $n = 0, \pm1, ...$ is an integer, and k_0 is the unstable wavenumber associated with a vacuum waveguide mode. Coupling to an infinite number of modes may be avoided in the approximation that $4K_d^2/k_q^2 \ll 1$. To zeroth order, the approximate dispersion relation is

$$T_0 = 0. \qquad (6)$$

To first and second order, we find

$$det\begin{pmatrix} T_{-2} & (K_d^4/2)S_{-3} \\ (K_d^4/2)S_{+1} & T_0 \end{pmatrix} = 0 \qquad (7)$$

and

$$det\begin{pmatrix} T_{-2} & (K_d^4/2)S_{-3} & 0 \\ (K_d^4/2)S_{+1} & T_0 & (K_d^4/2)S_{-1} \\ 0 & (K_d^4/2)S_{+3} & T_{+2} \end{pmatrix} = 0 \qquad (8)$$

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respectively.

II. Analytical Results

Much insight can be obtained from analytical decomposition of the approximate dispersion relation, Eq. (6). This can give us analytical expressions for the growth rates and instability boundaries in parameter space.

The dispersion relation can be rewritten with the current coupling terms grouped together in a term $\bar{\sigma}$,

$$W_0 W_{+1} W_{-1} \Pi = \bar{\sigma}, \tag{9}$$

where

$$W_m(\omega, k) = (\omega/c)^2 - (k + mk_q)^2 - \mu_{11}^2, \tag{10}$$

$$\Pi(\omega, k) = \alpha_0^2 \alpha_{+1}^2 \alpha_{-1}^2 - \frac{K_d^4}{2}\left[\alpha_{+1}^2 + \alpha_{-1}^2\right], \tag{11}$$

$$\alpha_m(\omega, k) = \omega/v_o - (k + mk_q), \tag{12}$$

and

$$\bar{\sigma} = k_b^2 \alpha_0^2 \alpha_{+1}^2 \alpha_{-1}^2 \left[k_b^4 + W_{+1}W_{-1} + W_0 W_{-1} + W_0 W_{+1}\right]$$
$$-\frac{1}{2} k_b^2 K_d^4 W_0 \left[\alpha_{+1}^2 W_{-1} + \alpha_{-1}^2 W_{+1}\right]. \tag{13}$$

The polynomial Π can be rewritten as

$$\Pi(\omega, k) = (\alpha_0 - k_q)^2 \Lambda - 2k_q K_d^4 \alpha_0, \tag{14}$$

where

$$\Lambda(\omega, k) = \left[\left(\alpha_0 + \frac{k_q}{2}\right)^2 - \left(\frac{k_q^2}{4} - K_d^2\right)\right]$$
$$\times \left[\left(\alpha_0 + \frac{k_q}{2}\right)^2 - \left(\frac{k_q^2}{4} + K_d^2\right)\right]. \tag{15}$$

Since K_d^4 is typically much less than 1, the last term in (14) can be dropped.

There are six waveguide modes, i.e., $W_0 W_{+1} W_{-1}$, and there are six beam modes. Four of the beam modes in (14) have the same characteristics as those in the helical quadrupole case.[2] As in the helical quadrupole case, the electron beam develops orbit instability, when

$$(k_q^2/4 - K_d^2) < 0, \tag{16}$$

with or without electromagnetic modes in the waveguide. Electromagnetic instability exists at the intersection of the

$$W_{-1}(\omega, k) = 0 \tag{17a}$$

waveguide mode and the

$$k + (-k_q/2 + \sqrt{k_q^2/4 - K_d^2}) - \omega/v_0 = 0 \tag{17b}$$

beam mode.

Following the same procedure as outlined in Ref. 2, analytical expressions for the spatial or temporal growth rates can be obtained. The derivation for the spatial growth rate is presented here. Electromagnetic instability occurs at frequency ω which satisfies

$$\sqrt{(\omega/c)^2 - \mu_{11}^2} + k_q = (\omega/v_0) + k_q/2 - \sqrt{k_q^2/4 - K_d^2}. \tag{18}$$

The dispersion relation (9) can be rewritten as

$$(k^2 - k_0^2)((k + k_q)^2 - k_0^2)((k - k_q)^2 - k_0^2)(k - k_2)^2$$
$$\times[(k - k_4)^2 - \Delta k_1^2][(k - k_4)^2 - \Delta k_2^2] \simeq -\bar{\sigma}, \tag{19}$$

where $k_0 = \sqrt{(\omega/c)^2 - \mu_{11}^2}$, $k_1 = \omega/v_0$, $k_2 = (\omega/v_0) - k_q$, $k_3 = (\omega/v_0) + k_q$, $k_4 = (\omega/v_0) + k_q/2$, $\Delta k_1 = \sqrt{k_q^2/4 - K_d^2}$, and $\Delta k_2 = \sqrt{k_q^2/4 + K_d^2}$. Defining $k = k_0 + k_q + \delta k$, the imaginary part of δk is the spatial growth rate. The analytical growth rate expression can be simplified to

$$\delta k^2 = \sigma / D_a, \tag{20}$$

where $\sigma = \bar{\sigma}|_{k=k_0+k_q}$,

$$\sigma \simeq k_b^2 (k - k_3)^2 (k^2 - k_0^2)((k + k_q)^2 - k_0^2)$$
$$[(k - k_1)^2 (k - k_2)^2 - \frac{K_d^4}{2}] \tag{21}$$

and

$$D_a \simeq -4\Delta k_1 (\Delta k_2^2 - \Delta k_1^2) k_0 (k - k_2)^2$$
$$\times (k^2 - k_0^2)((k + k_q)^2 - k_0^2)|_{k=k_0+k_q}. \tag{22}$$

When the more complete dispersion relation (8) is considered, growth rate at other intersections in (ω, k) of waveguide modes and other beam modes will collapse to an expression similar to Eq. (20) (with the wavenumber k shifted by $\pm nk_q$).

III. Examples

Here, the dispersion relations (8) and (9) are solved numerically. We verified that i) Eq. (9) is a fair approximation to (8), ii) the beam mode decomposition of (9) is accurate, iii) the boundary of orbit instability is as predicted by Eq. (16) and iv) analytical growth rate expressions are in good agreement with the results from (8).

Parameters for numerical examples are typical of a high-current, induction-accelerator beam, $I_e = 1\ kA$ and $\gamma_o = 7$. We chose a quadrupole gradient of $B_q k_q = 200$ G/cm, quadrupole wavenumber of $k_q = 0.5$ cm^{-1} and the waveguide radius of $r_g = 3\ cm$. Figure 1 plots the dispersion relation (9). The solid circle in Fig. 1 marks the instability denoted by Eqs. (17a,b). Figure 2 has plots of the spatial growth rate as a function of wavenumber, k, for $\omega > 0$ from the improved dispersion relation (8). The instability is periodic in k. The growth rates of all the instabilities are identical, except one. The instabilities at ($k \simeq k_0 - k_q$ and $k \simeq k_0 - 2k_q$) are from coupling to

T_{+2}. The instabilities from coupling to T_{-2} (not shown) are at ($k \simeq k_0 + 2k_q$ and $k \simeq k_0 + 3k_q$). The instability at ($k \simeq k_0 - 2k_q$), similar to the instability marked by the dashed circle in Fig. 1, is erroneous, and is modified to give the same growth rate as that at $k_0 + k_q$ as more terms of the full determinant (2) are included. The analytical result from Eq. (20) is indicated by the $\times$ on Fig. 2.

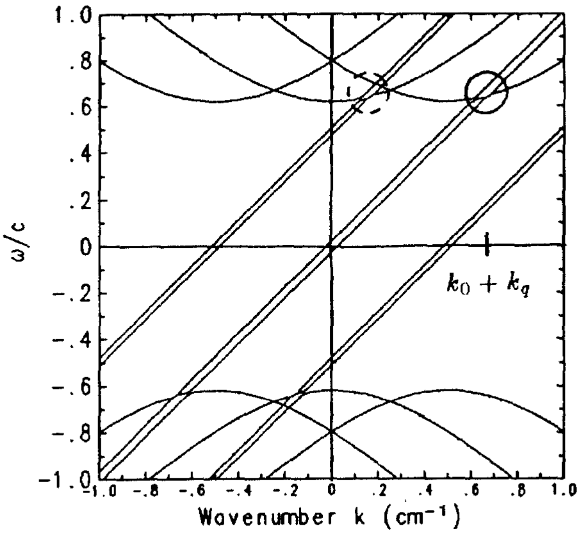

Fig. 1. Dispersion diagram of Eq. (9).

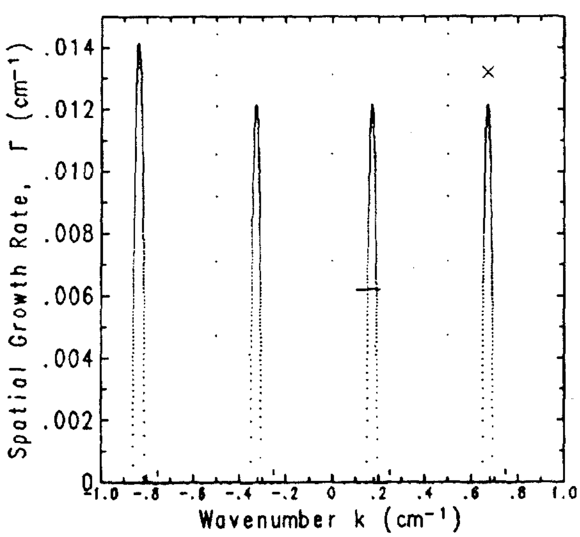

Fig. 2. Spatial growth rates from Eq. (8) for $\omega > 0$.

IV. Summary and Comments

Electromagnetic instability was found to grow on an intense beam in a FODO lattice. Exact and approximate dispersion relations were derived. Analytical growth rates were obtained and are in good agreement with numerical solutions of the dispersion relation. The boundary between electromagnetic and orbit instabilities was obtained and verified. The analysis will be extended to consider cases which include an additional nonzero axial (solenoidal) field.

Note that this instability has not been observed in rf linear accelerators because the electron beam pulse length is short. The instability group velocity is less than the speed of light and propagates out of the tail of the beam pulse before it can grow substantially.

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