Simulation of Hollow Beams with Cancellation of Steady State Non-linear Space-charge

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Abstract

This paper describes a technique for making multi-particle computer simulations of coherent instability consistent with conventional bunched-beam longitudinal instability theory. The argument behind the technique exposes an oversight of instability theory: the response of the phase-space distribution to the non-linear steady state wakefields is neglected. As an example, the technique is applied to a beam with the steady-state space-charge fields artificially cancelled. The computer simulations presented are seen to agree with mode-coupling theory. Beam and machine parameters are taken from the proposed TRIUMF-KAON [1] Accumulator, which is a high current proton storage ring.

I. Critique of Stability Analysis

The stability analysis of a particle beam is a two step process: (i) determine the steady state conditions; and (ii) discover the behaviour of small perturbations.

A. Steady State

The single particle equation of motion is:

$$\ddot{x} + \omega_s^2 x = F(x) = \omega_s^2 \xi \sum_p Z_p(0) \Lambda_p e^{jpx} \quad . \tag{1}$$

The wakefield F is the product of the complex impedance $Z_p(0) = Z(ph\omega_0)$ evaluated at frequencies $ph\omega_0$ and the steady state beam Fourier components Λ_p. Here ω_0 is the revolution frequency. The wakefield F comprises a linear part $L(x)$ and a non-linear part $N(x)$. The linear term is removed by renormalizing the incoherent frequency. Then (1) becomes:

$$\ddot{x} + \langle \omega_{\rm inc}^2 \rangle x = N(x) \quad . \tag{2}$$

Here $\langle \omega_{\rm inc} \rangle$ is the ensemble average incoherent frequency.

The behaviour of the steady state ensemble with phase-space distribution function $\Psi(r,\theta)$ is governed by the Vlasov equation. This is written in polar coordinates:

$$\sqrt{\langle \omega_{\rm inc}^2 \rangle} \frac{\partial}{\partial \theta} \Psi \; - \; N(x) \left[\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r} \frac{\partial}{\partial \theta} \right] \Psi = 0 \quad . \tag{3}$$

If $N \equiv 0$ then Ψ has no angular dependence and so $\Psi = \Psi(r)$. We can turn the argument around, and say that if the steady state distribution is a function only of radius, then the non-linear part of the wakefield must have been exactly cancelled by an externally applied counter-field. This is the assumption made in analytical instability theory. Physically, the cancellation is very difficult to achieve; but it is trivial to implement in a computer simulation.

B. Small Perturbations

We write the time dependent part of the phase-space ensemble as $\varepsilon\psi(r,\theta)e^{j\omega t}$; and this generates a wakefield

$$\varepsilon f(t,x) = \varepsilon \; e^{j\omega t} \sum_p Z_p(\omega) \, \lambda_p \, e^{jpx} \tag{4}$$

where λ_p are the harmonics of the perturbation bunch-shape and $Z_p(\omega) = Z(ph\omega_0 + \omega)$ is the impedance at the perturbation frequency ω. The Vlasov equation becomes:

$$\left[\frac{\partial}{\partial t} - \langle \omega_{\rm inc} \rangle \frac{\partial}{\partial \theta} \right] \left[\Psi + \varepsilon \psi e^{j\omega t} \right] \; + $$
$$\frac{\varepsilon f(x,t)}{\langle \omega_{\rm inc} \rangle} \left[\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r} \frac{\partial}{\partial \theta} \right] \left[\Psi + \varepsilon \psi e^{j\omega t} \right] = 0 \quad . \tag{5}$$

(5) is linearized in ε, and the steady state equation (3) is subtracted giving:

$$\left[j\omega - \langle \omega_{\rm inc} \rangle \frac{\partial}{\partial \theta} \right] \psi \; + \; \frac{f(x)}{\langle \omega_{\rm inc} \rangle} \sin\theta \frac{\partial}{\partial r} \Psi(r) = 0 \quad . \tag{6}$$

C. Consistency Technique

Instability theory seeks solutions of (6). A multi-particle simulation code is easily made to emulate equation (5), and this is a close relative of equation (6). We simply subtract an analytic expression for the steady state wakefield from the wakefield calculated from the Monte Carlo ensemble. When this is done, we should expect agreement on the threshold beam current for coherent instability; because the simulation has been made as unrealistic as the analytic instability theory.

D. Real Beam

Of course, the real beam *will* respond to the steady state wakefield. When the non-linear fields are substantial, then complicated mismatching effects will occur; and though they may mimic an instability these are, in fact, transient response behaviours. In this light, it is no surprise that the calculations and simulation for a space-charge dominated hollow beam reported in reference [2] differ in their conclusions.

II. Mode Coupling Results

An instability analysis for a bunched beam which is hollow in longitudinal phase-space and for which the internal self-forces derive solely from the space-charge force has been carried out by Baartman [2]. Mode-coupling theory shows

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there to be a dipole instability when the zero amplitude incoherent frequency $\omega(0)$ becomes $\sqrt{2}$ times the zero intensity synchrotron frequency ω_s, as in figure 3.

A further prediction of reference [2] is that there is no quadrupolar instability until the threshold frequency $\omega(0) \approx 1.58\omega_s$; also as in figure 3.

III. Simulation Model

The computer experiments were made with the code LONG1D [3]. The rf-cavities produce a linear restoring force. There are 4 cavity crossings per turn, and space-charge is implemented by a second order symplectic mapping with 4 sub-steps between each cavity. The beam bunch is modelled by an ensemble of 6×10^4 macro-particles. A random number generator was used to prepare an approximation to the hollow gaussian distribution:

$$\Psi(r) = (r^2/4\pi\sigma^4)\exp(-r^2/2\sigma^2) \ . \qquad (7)$$

The parameter σ is measured in radians of rf-phase, and the spacing between bunches is 2π. The finite number of simulation particles implies a statistical jitter in the bunch shape, and this is responsible for seeding any unstable behaviour that may occur.

A. *Cancellation of Steady State Wake*

The steady state Fourier components are:

$$2\pi\Lambda_p \ = \ \left[1 - p^2\sigma^2/2\right]\exp(-r^2\sigma^2/2) \qquad (8)$$

and these are subtracted from the components found by binning the statistical ensemble and Fourier analysing. The remainder gives the residual λ_p.

The linear part of the wakefield is restored by substituting the ensemble average incoherent tune in place of the zero intensity synchrotron tune.

$$\langle\omega_{\rm inc}^2\rangle \ = \ \omega_s^2\left[1 - C/16\sigma^3\sqrt{\pi}\right] \qquad (9)$$

The bunch length parameter σ has to be adjusted to account for the bunch lengthening consistent with $\langle\omega_{\rm inc}\rangle$. Let β, γ be relativistic kinematic parameters; Z_0 the impedance of free space; $\hat{V}_{\rm rf}$ the peak accelerating voltage; $\langle I\rangle$ the dc component of circulating current in Amps; h the harmonic number; and $g_0 = 1 + 2\ln(b/a)$. The constant C is given by

$$C \ = \ \left(\frac{g_0 Z_0 h}{2\beta\gamma^2}\right)\left(\frac{2\pi\langle I\rangle}{\hat{V}_{\rm rf}\cos\phi_s}\right) \ . \qquad (10)$$

Equations (7–11) are taken from reference [4].

IV. Ensemble Characteristics

The behaviour of the beam coherent motion was monitored by extracting 2 orthogonal measures of the dipole moment, $\langle r\cos\theta\rangle$ and $\langle r\sin\theta\rangle$; and 2 measures of the quadrupole moment, $\langle r^2\cos(2\theta)\rangle$ and $\langle r^2\sin(2\theta)\rangle$. The quadrature sum of 2 measures will give the net amplitude of the dipolar or quadrupolar disturbance. The Fourier transform with respect to time of an individual measure will show the frequencies of the many coherent radial modes which contribute to each polar disturbance.

A. *Ellipse Rotation and Scaling*

When longitudinal motion is modelled by difference equations (rather than a differential equation) the matched phase-space circle becomes a tilted ellipse. Whereas the polar moments of a uniformly filled circle are zero, those of an ellipse are finite. Consequently, the ensemble coordinates have to be multiplied by the inverse longitudinal Twiss parameter matrix before the multipole moments are extracted. This procedure is described in reference [5].

V. Trial Cases And Results

Following Baartman, the trial cases are ordered as a function of the zero amplitude incoherent frequency

$$\omega^2(0) = \omega_s^2\left[1 + C/2\sigma^3\sqrt{2\pi}\right] \ . \qquad (11)$$

Trial cases for a space-charge compensated beam are summarized in Table 1. The entries in the last column indicate if the simulation behaved as below ($\Downarrow$), or at dipolar threshold ($\rightleftharpoons$), or above ($\Uparrow$). The beam current is given in Amps.

Table 1: Trial cases for beam with hollow phase-space.

Case	Current	σ	$\langle\omega_{\rm inc}\rangle/\omega_s$	$1-\omega^2(0)/\omega_s^2$	
A	0.00	0.500	1.00	0.00	n/a
B	1.5867	1.00256	0.99489	−0.05993	$\Downarrow$
C	4.4008	1.00708	0.98599	−0.16399	$\Downarrow$
D	2.2004	0.51459	0.94411	−0.61462	$\Downarrow$
E	3.6673	0.52407	0.91026	−0.96976	$\Downarrow$
F	4.0341	0.52641	0.90218	−1.05257	$\rightleftharpoons$
G	4.4008	0.52874	0.89425	−1.13315	$\Uparrow$
H	5.1343	0.53336	0.87882	−1.28795	$\Uparrow$
I	7.3347	0.54695	0.83569	−1.70617	$\Uparrow$

A. *Dipolar Instability*

Figure 1 shows the dipolar amplitude versus time for 3 cases (G, H, I) above threshold and one case (E) below threshold. The threshold frequency shift $\omega(0) \approx 1.4327\omega_s$, inferred from case F, is in excellent agreement with the value predicted by Baartman, $\omega(0) \approx 1.4142\omega_s$.

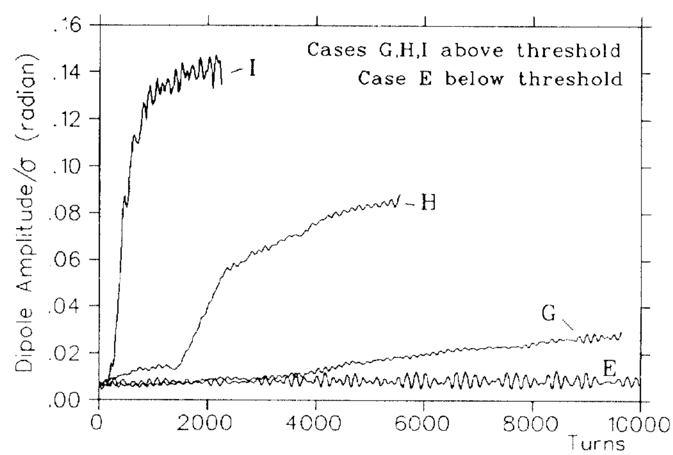

Fig.1 Space-charge compensated hollow beams

Well above threshold, as in cases H and I, the effect of the instability is to transform the initial two-humped line density into a single-humped gaussian distribution.

B. Quadrupolar Instability

There should be no growth in the quadrupolar amplitude for cases A through H. However, for trial case I the zero amplitude frequency is $\omega(0) = 1.645\,\omega_s$; well above threshold and quadrupolar growth is expected. Figure 2 shows the quadrupolar amplitude versus time for cases E, G, H and I; confirming the predictions for the quadrupolar instability threshold.

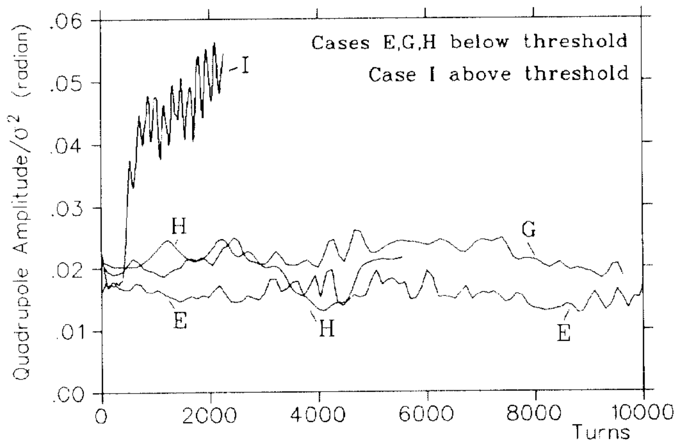

Fig.2 Space-charge compensated hollow beams

C. Frequency Spectra

The theoretical frequency spectrum of coherent oscillation modes is given in figure 3, which is adapted from reference [2]. The precise values and number of modes depends on the size of the matrix used to find the eigen-frequencies. Each polar disturbance is split into several radial modes.

It is anticipated that the fast Fourier transform (FFT) of the simulation polar components should show a similar pattern of frequencies. This is verified in the example figures 4a and 4b. The ordinate ω_0/ω is the inverse of the synchrotron tune. The small amplitude zero-intensity tune is $\omega_s/\omega_0 = .04883$. The annotations give the oscillation frequency divided by the ensemble average incoherent synchrotron frequency. The amplitude of a particular frequency component derives from statistics of the ensemble and how well the polar measures discriminate a particular radial mode. Consequently, not all radial mode frequencies need be present in the FFT.

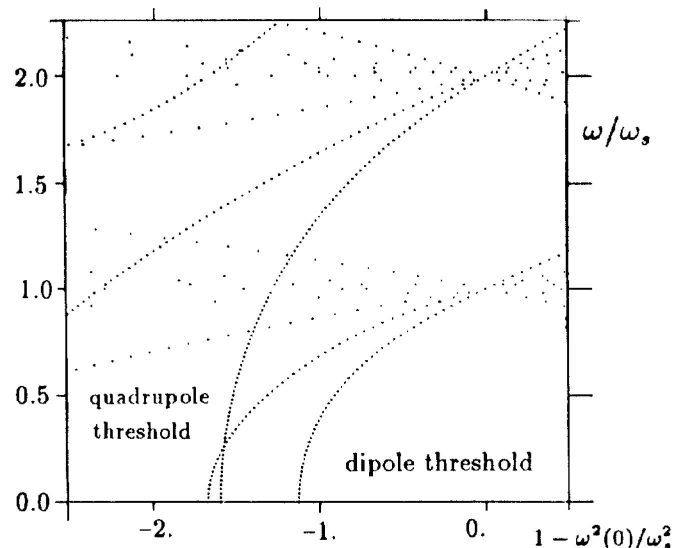

Fig.3: Theoretical frequency spectrum.

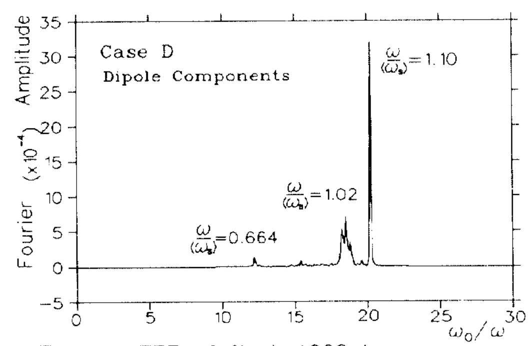

Fig.4a FFT of first 4000 turns

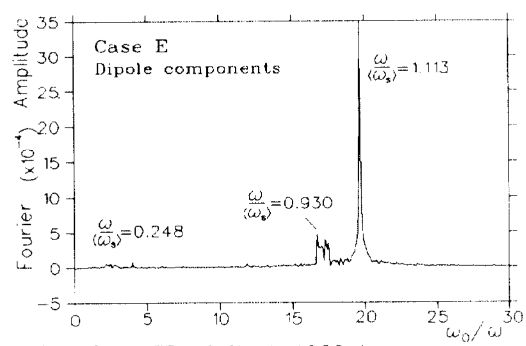

Fig.4b FFT of first 4000 turns

VI. Conclusion

Bunched beam instability theory applies only to a beam in which the non-linear steady state wakefields have been artificially cancelled. Consistency between simulation and calculation demands that a similar, albeit non-physical, compensation of steady state wakes is incorporated into multi-particle codes. When this is done, the agreement between the simulations of a space-charge compensated beam presented here, and the analysis of Baartman leads to a high level of confidence in the computer model.

VII. References

[1] Kaon Factory Study, Accelerator Design Report; TRIUMF, Vancouver, Canada, March 1990.

[2] R. Baartman and S. Koscielniak : "Stability of Hollow Beams in Longitudinal Phase Space"; Particle Accelerators, 1990, Vol.28 pp.95–99

[3] S. Koscielniak: "The LONG1D Simulation Code", Proc. *1st European Particle Accelerator Conference*, Rome 1988, pg. 743.

[4] S. Koscielniak: "Phase and incoherent frequency shifts due to steady-state wakefields"; TRI-DN-90-K136.

[5] S. Koscielniak: "Detailed Longitudinal Phase-Space Matching"; TRI-DN-91-K160.