PERTURBATION TREATMENT OF THE LONGITUDINAL COUPLING IMPEDANCE OF A TOROIDAL BEAM TUBE*

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Abstract

A simple analytical expression for the longitudinal coupling impedance of a toroidal beam tube below the resonance region has been derived by expanding the electromagnetic fields of the toroidal beam tube in a power series in curvature and substituting directly into Maxwell's equations. The resulting expression consists of the impedance of the straight beam pipe plus a correction terms due to the curvature. It has been verified that this result gives excellent agreement to the exact solution below the first resonance.

INTRODUCTION

The longitudinal coupling impedance of a toroidal beam tube with rectangular cross section was addressed by a number of recent publications.[1] An exact treatment of this problem with many references to earlier work can be found in the paper by Warnock and Morton.[2]

An approximate expression for the coupling impedance below the resonance region, but valid beyond cut off, was derived by Ng and Warnock using the Debye's asymptotic expansions of relevant Bessel functions.[3] In the present paper, a simple analytical expression for the curvature term of the sub-resonant coupling impedance is derived by expanding the electromagnetic fields in a power series in curvature of the toroidal beam tube and substituting directly into Maxwell's equations.

The perturbation treatment of electromagnetic problems is well-known due to the work by Jouguet[4] and Lewin[5] and its application to the present problem conveys considerable physical insight without loss of accuracy. In fact, it has been computationally verified that the results presented here are in better agreement with the exact solution than the Ng–Warnock approximation, although the differences are inconsequential.

FORMAL SOLUTION

A convenient method of obtaining an expression for the longitudinal coupling impedance presented by a smooth beam tube with rectangular cross section (Fig. 1) to a filamentary current involves field matching along a vertical plane common to the inner and outer subregion. In the case of a curved tube, the fields must be expanded in terms of H and E modes. In a straight tube, a pure TM mode would be adequate. A formal solution, valid for both cases, is obtained by always using H/LSE and E/LSH modes.

The centered filamentary current $Ie^{-jn\theta}e^{j\omega t}$ (mode number n, frequency $\omega = vn/R$) is represented by a current sheet in the vertical plane,

$$i = \frac{2I}{h} e^{-jn\theta} e^{j\omega t} \sum_m \cos \xi_m z$$

with $\xi_m = m\pi/h$; m = odd. To achieve notational simplicity, the index m and the common exponential factor will be suppressed in the sequel.

The field components in each subregion have the general form

$$\mathbf{E} = \begin{pmatrix} \mathcal{E}_r \cos \xi z \\ j\mathcal{E}_\theta \cos \xi z \\ \mathcal{E}_z \sin \xi z \end{pmatrix} \text{ and } \mathbf{H} = \begin{pmatrix} \mathcal{H}_r \sin \xi z \\ j\mathcal{H}_\theta \sin \xi z \\ \mathcal{H}_z \cos \xi z \end{pmatrix}$$

The field in each subregion is given by the sum of E and H modes with 2 expansion coefficients (per index m). The four coefficients are determined by requiring continuity of E_θ, E_z, H_θ and application of Ampère's law to H_z at the vertical current sheet. The longitudinal coupling impedance is defined by

$$\frac{Z}{n} = -\frac{R}{nI} \int_0^{2\pi} E_\theta \, (z = 0, x = 0) \, e^{jn\theta} d\theta$$

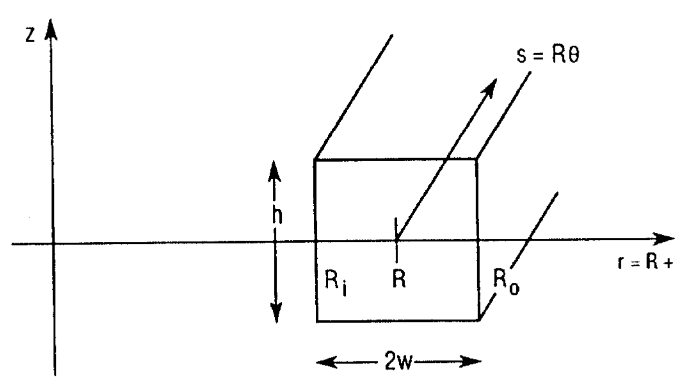

Fig. 1: Toroidal Beam Tube Geometry.

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Table I: Field Components in Beam Tube with Rectangular Cross Section.

H(LSE) mode toroidal	straight	E(LSH) mode toroidal	straight
$\mathcal{E}_r = v\Omega^2 \frac{R}{r} C(\kappa r)$	$v\Omega^2 \cosh \chi (x \pm w)$	$-\frac{\kappa}{\xi} S'(\kappa r)$	$-\frac{\chi}{\xi} \cosh \chi (x \pm w)$
$\mathcal{E}_\theta = -v\Omega \frac{\kappa}{\xi} C'(\kappa r)$	$-v\Omega \frac{\chi}{\xi} \sinh \chi (x \pm w)$	$\Omega \frac{R}{r} S(\kappa r)$	$\Omega \sinh \chi (x \pm w)$
$\mathcal{E}_z = 0$	0	$(1 - v^2\Omega^2) S(\kappa r)$	$(1 - v^2\Omega^2) \sinh \chi (x \pm w)$
$\mathcal{H}_r = \frac{\kappa}{\xi} C'(\kappa r)$	$\frac{\chi}{\xi} \sinh \chi (x \pm w)$	$-v\Omega^2 \frac{R}{r} S(\kappa r)$	$-v\Omega^2 \sinh \chi (x \pm w)$
$\mathcal{H}_\theta = -\Omega \frac{R}{r} C(\kappa r)$	$-\Omega \cosh \chi (x \pm w)$	$v\Omega \frac{\kappa}{\xi} S'(\kappa r)$	$v\Omega \frac{\chi}{\xi} \cosh \chi (x \pm w)$
$\mathcal{H}_z = (1 - v^2\Omega^2) C(\kappa r)$	$(1 - v^2\Omega^2) \cosh \chi (x \pm w)$	0	0

with the formal solution in natural units ($c = \mu_o = 1$)

$$\frac{Z}{n} = j\frac{2}{v} \sum \frac{1}{m\Omega (1 - v^2\Omega^2)} \times$$
$$\times \left\{ \frac{2\mathcal{E}^E_{\theta i}\mathcal{E}^E_{\theta o}}{\mathcal{E}^E_{ri}\mathcal{E}^E_{\theta o} - \mathcal{E}^E_{\theta i}\mathcal{E}^E_{ro}} - \frac{2v^2\Omega^2\mathcal{H}^H_{ri}\mathcal{H}^H_{ro}}{\mathcal{H}^H_{\theta i}\mathcal{H}^H_{ro} - \mathcal{H}^H_{ri}\mathcal{H}^H_{\theta o}} \right\}$$

where

$$\Omega = \frac{n}{\xi R} = \frac{n}{m\pi} \frac{h}{R}$$

and the field components $\mathcal{E}$ and $\mathcal{H}$ for inner (index i) and outer (index o) subregion to be evaluated at $x = 0$. The present solution equals in substance the Warnock–Morton result, however the different formulation is essential to the subsequent treatment of the problem. Note also, that the distinction between $Z(n)$ and $Z(n,\omega)$ can be ignored in the subresonant region, which is addressed in this paper.

The expressions for the field components of E and H modes are given in Table I. In the toroidal case the functions $C(\kappa r)$ and $S(\kappa r)$ are linear combinations of modified Bessel functions with their definition given in Table II and

$$\kappa = \xi\sqrt{1 - v^2\Omega^2}.$$

In the straight case

$$\chi = \xi\sqrt{1 + \Omega^2/\gamma^2}$$

with the relativistic $\gamma^{-2} = 1 - v^2$. Note that the boundary at conditions R_i and R_o, respectively $x = \pm w$, are fully satisfied.

The coupling impedance of the straight beam tube, the space charge term, follows as ($Z_0 = c\mu_0 = 1$)

$$\frac{Z}{n} = -jZ_0 \frac{2}{v\gamma^2} \sum \frac{\mathrm{tgh}\, \xi w\sqrt{1 + \Omega^2/\gamma^2}}{m\sqrt{1 + \Omega^2/\gamma^2}}$$

which vanishes for $v \sim 1$.

PERTURBATION TREATMENT

The residual curvature term for $v \sim 1$ could be obtained from the above formal solution by using asymptotic expansions for the Bessel functions similar to the Ng–Warnock treatment.[3] An asymptotic expansion is here obtained directly from Maxwell's equations by expanding the field components according to

$$\mathcal{E} = E(x) + e(x)/R$$
$$\mathcal{H} = H(x) + h(x)/R$$

with the zero order solution given by the straight tube. The resulting set of differential equations is given in Table III. Higher-order solutions are obtained by iteration.

Using the symbolic manipulation program MACSYMA, asymptotic expressions for the field components were determined to second order in w/R for the case of $v \sim 1$. The results for $C_i(\kappa R)$ and $S_i(\kappa R)$ are given as example in Table II.

Table II: Definition and Asymptotic Expansion of Toroidal Functions.

$$C_{i,o}(\kappa r) = \kappa\sqrt{RR_{i,o}} \left\{K_n(\kappa r) I'_n(\kappa R_{i,o}) - I_n(\kappa r) K'_n(\kappa R_{i,o})\right\}$$
$$C_i(\kappa R) \sim \cosh \xi w +$$
$$+ \frac{w}{R} \left\{ \frac{1-(1-\xi^2w^2)\Omega^2}{2\xi w} + \frac{w}{R} \frac{\xi w(3-\Omega^2)\Omega^2}{24} + \frac{3-10\Omega^2+7\Omega^4}{32\xi R} \right\} \sinh \xi w + \frac{w^2}{R^2} \left\{ \frac{(1-\Omega^2)\Omega^2}{32} + \frac{3-10\Omega^2+7\Omega^4}{64\xi^2w^2} \right\} \cosh \xi w$$
$$S_{i,o}(\kappa r) = \kappa\sqrt{RR_{i,o}} \left\{I_n(\kappa r) K_n(\kappa R_{i,o}) - K_n(\kappa r) I_n(\kappa R_{i,o})\right\}$$
$$S_i(\kappa R) \sim \sinh \xi w +$$
$$+ \frac{w}{R} \left\{ \frac{\xi w\Omega^2}{2} + \frac{w}{R} \frac{\xi w(3-\Omega^2)\Omega^2}{24} - \frac{1-6\Omega^2+5\Omega^4}{32\xi R} \right\} \cosh \xi w + \frac{w^2}{R^2} \left\{ \frac{(1-\Omega^2)\Omega^2}{32} + \frac{1-6\Omega^2+5\Omega^4}{64\xi^2w^2} \right\} \sinh \xi w$$

Table III: Perturbation Formulation of Maxwell's Equations.

$$\frac{\partial^2}{\partial x^2}h_z - \left(\xi^2 - \omega^2 + \frac{n^2}{R^2}\right) h_z = +\frac{\xi^2-\omega^2}{\omega}\left(x\frac{\partial}{\partial x}E_\theta + \omega x H_z + E_\theta\right) - \frac{\xi}{\omega}\frac{\partial}{\partial x}\left(\xi x E_\theta + \omega x H_r\right) + \frac{n}{R}\left(\xi x H_\theta + \omega x E_r\right)$$
$$\frac{\partial^2}{\partial x^2}e_z - \left(\xi^2 - \omega^2 + \frac{n^2}{R^2}\right) e_z = -\frac{\xi^2-\omega^2}{\omega}\left(x\frac{\partial}{\partial x}H_\theta - \omega x E_z + H_\theta\right) + \frac{\xi}{\omega}\frac{\partial}{\partial x}\left(\xi x H_\theta + \omega x E_r\right) - \frac{n}{R}\left(\xi x E_\theta + \omega x H_r\right)$$
$$h_r = \frac{\omega}{\xi^2-\omega^2}\left(-\frac{n}{R}e_z + \frac{\xi}{\omega}\frac{\partial}{\partial x}h_z + \xi x E_\theta + \omega x H_r\right)$$
$$e_r = \frac{\omega}{\xi^2-\omega^2}\left(\frac{n}{R}h_z - \frac{\xi}{\omega}\frac{\partial}{\partial x}e_z + \xi x H_\theta + \omega x E_r\right)$$
$$h_\theta = -\frac{1}{\omega}\left(\xi e_r + \frac{\partial}{\partial x}e_z\right)$$
$$e_\theta = -\frac{1}{\omega}\left(\xi h_r - \frac{\partial}{\partial x}h_z\right)$$

The perturbation expression for the curvature term of the longitudinal coupling impedance of a toroidal beam tube with rectangular cross section was found to be

$$\frac{Z}{n} = -jZ_0\left(\frac{h}{\pi R}\right)^2 \sum \left(1 - 3\Omega^2\right)\frac{\frac{1}{2}\sinh 2\xi w - \xi w}{m^3\cosh^2\xi w}$$

This perturbation expression has been numerically compared with the exact results obtained by using the SLATEC subroutines for the Bessel functions. It was confirmed that the total coupling impedance in the relativistic case of $v \sim 1$ is given by the sum of the straight beam tube space charge term plus the γ-independent curvature term.[3]

In Fig. 2, the total Z/n from above approximations is compared with the exact results as function of γ for a geometry with $2w = h$ and $\pi R/h = 10^4$. It is noted that the modified Bessel functions revert to ordinary Bessel functions at $\Omega = v^{-1} \sim 1$, the cut-off frequency corresponds to $\Omega \approx \sqrt{2}$ and the first resonance occurs at $\Omega_s = 85.8825$. Complete agreement at frequencies up to and above cut-off, but below the resonance region, was found.

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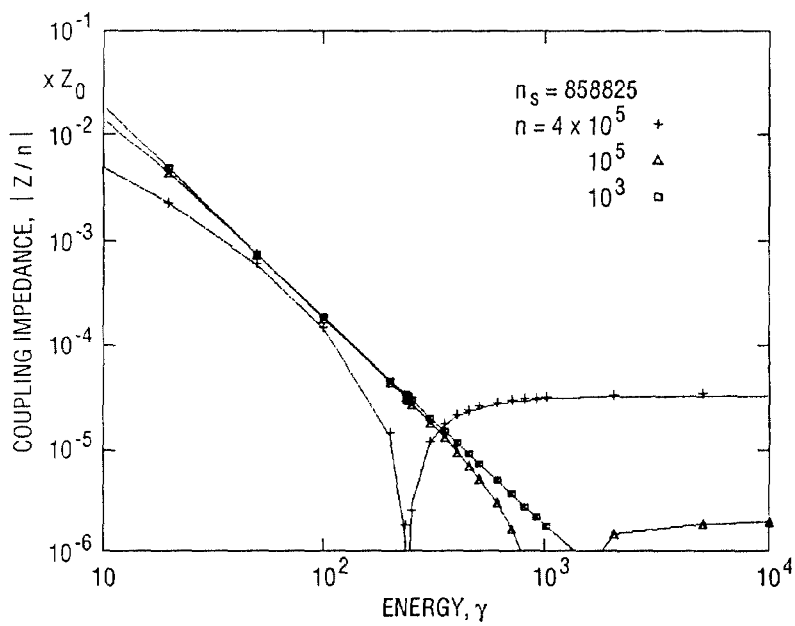

Fig. 2: Coupling impedance as function of γ. The solid curves are obtained from the approximations in this paper and the points from the exact expressions.