Emittance Calculation Using Liouvilles Theorem for a Diagonalized Hamiltonian

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Abstract

Application of Liouville theorem for the calculation of emittance of particle beams through a special electrostatic arrangement using the separation of phase space by means of a diagonalizable structure of the underlying Hamiltonian.

I. Introduction

It is demonstrated, that in a special electrostatic arrangement the z-dependence (direction of beam) of the Hamiltonian can be neglected. The Hamiltonian separates into two parts describing the degrees of freedom of the perpendicular motion of the particles. This allows using Liouville's theorem to calculate emittance. A formula is easily derived by taking into acount the additive structure of the Hamiltonian (constant partial phase space). A simple method is presented for measuring the emittance.

II. Liouville theorem and the constancy of the phase space

Ions represent a system consisting of n particles corresponding to f degrees of freedom in the phase space. Hence their total mechanical state is described by $F = nf$ generalized configuration coordinates and F generalized conjugate impulse coordinates, therefore by a point in a $2F$-dimensional space in which the fundamental laws of mechanics read:

$$\dot{\vec{R}} = \omega \frac{\partial}{\partial \vec{R}} H \ , \ \omega = \begin{bmatrix} 0 & +1_{f\times f} \\ -1_{f\times f} & 0 \end{bmatrix} \tag{1}$$

$H = H(q^i, p_k)$ is the Hamiltonian,

$$\vec{R} = [q^1 \ldots q^f \ p^1 \ldots p^f]^T \quad \in R^{2q} \quad , \tag{2}$$

$\dot{q}_j = -\partial H/\partial q_j$ and $\dot{p}_j = \partial H/\partial p_j$ are canonical conjugated variables. Given $H = H(\vec{R}, t)$ from (1) follows

$$\omega \frac{\partial}{\partial \vec{R}} H(\vec{R}, t) = \vec{V}^H(\vec{R}, t) \quad , \tag{3}$$

where $\vec{V}^H$ is the Hamiltonian vector field representing the time derivative along the trajectories in the phase space. Due to the fact that the Hamiltonian vector field $\vec{V}^H$ has always vanishing divergence in phase space R^{2f} for arbitrary phase space function $F = F(\vec{R}, t)$, it follows

$$\frac{\partial}{\partial \vec{R}} \vec{V}^H(\vec{R}, t) = 0 \quad . \tag{4}$$

By using Reynold's transport theorem and Gauss theorem results Liouville equation [1]:

$$\frac{\partial \varrho}{\partial t} = \vec{V}^H(\vec{R}, t) \frac{\partial}{\partial \vec{R}} \varrho(\vec{R}, t) = -\frac{\partial \varrho}{\partial \vec{R}} \omega \frac{\partial H}{\partial \vec{R}} = \{\varrho, H\} \ , \tag{5}$$

$$\frac{\mathrm{d}\varrho}{\mathrm{d}t} = 0$$

$\varrho(\vec{R}, t)\, \mathrm{d}^N \vec{R}$ permits the probability of finding N particles in the volume of the phase space $\mathrm{d}^N \vec{R}$ at time t. Now eq.(5) for 3 degrees of freedom is represented in the following form

$$\iiiint\!\!\iint \varrho(q_1, q_2, q_3, p_1, p_2, p_3)\, \mathrm{d}q_1\, \mathrm{d}q_2\, \mathrm{d}q_3\, \mathrm{d}p_1\, \mathrm{d}p_2\, \mathrm{d}p_3 = $$
$$= \mathrm{const.} \tag{6}$$

The local distribution, i.e. angular and energy distribution, immediately results from the angular distribution of the degrees of freedom, corresponding to each of the 3 space axis, the coordinates x, y and z, and the classical impulses $m\dot{x}$, $m\dot{y}$ and $m\dot{z}$.

Assuming the following that x, p_x and y, p_y are the conjugate canonical variables, secondly that the total energy and the impulse remains constant in z-direction and thirdly $\varrho(x, p_x, y, p_y)$ factorizes as follows $\varrho(x, p_x) \cdot \varrho(y, p_y)$ in the (x, y)-plane, the invariants

$$\iint \varrho(x, p_x)\, \mathrm{d}x\, \mathrm{d}p_x = const., \iint \varrho(y, p_y)\, \mathrm{d}y\, \mathrm{d}p_y = const. \tag{7}$$

can be derived by means of (5). But the classical impulses $m\dot{x}$ and $m\dot{y}$ are not always canonical impulses conjugate to the classical coordinates x and y.

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Moreover, eq.(2) shows that the conjugate canonical variables enter into the general expression of the Hamilton function

$$H = \sum_i p_i \dot{q}_i - L \tag{8}$$

related to the corresponding Lagrange function

$$L = T(q_i, \dot{q}_i) - \Pi(q_i)$$

where e.g. L could have the form [2]

$$L = -mc^2\sqrt{1-\beta^2} + e(\vec{v}\cdot\vec{A}) - e\Phi \tag{9}$$

describing the motion of a charged particle in electromagnetic fields with Π = potential energy, T = kinetic energy, Φ = scalar potential, $\vec{A}$ = vector potential. Example (9) shows that the presence of the term $(\vec{v}\cdot\vec{A})$ leads to a more complicated relation between the canonical impulses p_x, p_y (conjugate to x, y) and the 3 classical impulses $m\dot{x}$, $m\dot{y}$, $m\dot{z}$ thus preventing us from specifying invariants similar to (7).

Supposing that the total Hamiltonian splits additively [1]

$$H(\vec{R}_{(1)}, \vec{R}_{(2)}) = H_1(\vec{R}_{(1)}) + H_2(\vec{R}_{(2)}) \tag{10}$$

where $\vec{R}_{(1)}$ and $\vec{R}_{(2)}$ are collections of conjugate canonical variables. The time derivative of the projected volume of the phase space is given as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int\limits_{p_r(g(t))_{N_1}} \mathrm{d}^{N_1}\vec{R}_{(1)}\right] = 0 \quad . \tag{11}$$

The integration in eq.(11) applies to the projection of the total phase space domain $g(t)$ to the corresponding plane. In addition, the following conditions must be fulfilled

$$g(t) \subset R^N \ , \ \vec{R}_{(1)} \in R^{N_1} \ , \ \vec{R}_{(2)} \in R^{N_2} \ , \ N_{(1)} + N_{(2)} = N \quad .$$

The validity of (11) can be proven in analogy to (4), for example for the projection in the y, p_y-plane, as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int\limits_{p_r(g(t))_{(y,p_y)-plane}} \mathrm{d}y\,\mathrm{d}p_y\right] = 0$$

To achieve (7) appropriate coordinates with the property

$$\iint \varrho(z, p_z)\,\mathrm{d}z\,\mathrm{d}p_z = const. \tag{12}$$

are introduced, so that from equ.(12)

$$\iiiint \varrho(x, p_x, y, p_y)\,\mathrm{d}x\,\mathrm{d}p_x\,\mathrm{d}y\,\mathrm{d}p_y = const. \tag{13}$$

is derived, where the z-component of the impulse of the particle is assumed to be constant.

III. The emittance space and the emittance area

In the previous chapter it is demonstrated that the equation of motion of the trajectories (with time independent fields), relating six independent variables is overdetermined, one of them beeing taken as a parameter.

Now starting from the general form of the canonical conjugated coordinates $p_i = \mathrm{d}q_i/\mathrm{d}t$ the corresponding impulse in the x-coordinate

$$p_x = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}z}{\mathrm{d}t}\frac{\mathrm{d}x}{\mathrm{d}z} = p_z\,x' \approx p_0\,x' \tag{14}$$

is derived in a field free space. When the requirement, which has already been substantiated, $p_z = p_0 = const. = 1$ is inserted, then x' is the canonical coordinate conjugated to x and has the meaning of an angle projection onto the (x, z)-plane. This angle measures then deviation of the path of the particle from the z-axis. Therefore, the (x, p_x)-subspace of the phase space can be replaced by the (x, x')-space and respectively also (y, p_y)-subspace can be replaced by (y, y')-space thus leading to the emittance spaces E_x, E_y.

Fronteau called such spaces "double degenerated phase space" because of $p_z = p_0, p_0 = 1$. Explicitly stated, E_x coordinized by x and x' the emittance space for the x-direction and E_y (coordinized by y and y') for the $y-$direction respectively. The area that arises from both emittance spaces E is named emittance area $F_{xx'}$ and $F_{yy'}$, see also Fig.1d.

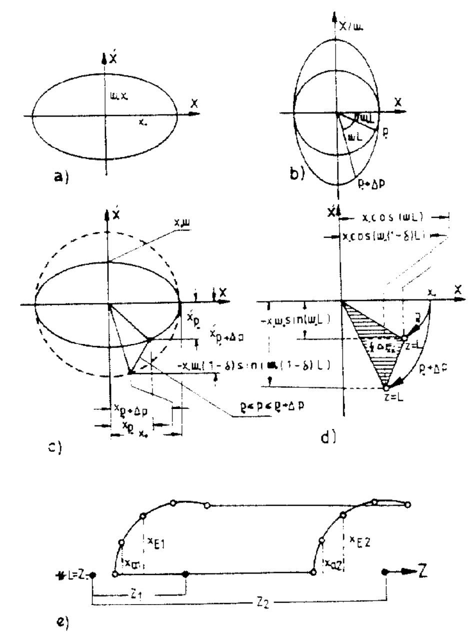

Figure 1: Schematic representation of the emittance area (a-d) and for measuring the emittance (e)

IV. Electrostatic arrangement and emittance area

For the potential of an electrostatic arrangement consisting of three lenses one can use the approach [1]

$$V = \frac{3}{8}\, U\, A\, \frac{\varrho^2}{R^3} \quad , \quad A \ll R \quad . \tag{15}$$

There are U - voltage between two lenses, A - distance between them, R - radius of iris and ϱ - component of cylindrical coordinates. Now the potential is z-independent.

The corresponding Hamilton function of a particle in this system reads

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + q\, U\, A\, \frac{1}{R^3}(x^2 + y^2)\frac{3}{8} \quad . \tag{16}$$

Eq.(16) shows that the canonical impulse conjugated to x is independent of y. Therefore the equation of motion in any of those directions is dependent only on its respective direction coordinates and not on the other coordinates. Eq.(16) permits to derive the subsequent equations of motion

$$\frac{\mathrm{d}^2 x}{\mathrm{d}z^2} + \omega_0^2 x = 0 \ , \ \frac{\mathrm{d}^2 y}{\mathrm{d}z^2} + \omega_0^2 y = 0$$
$$\omega_0^2 = \frac{3}{4}\frac{qUAm}{R^3 p_z^2} \ , \ p_z = p_0 \quad . \tag{17}$$

In order to examine the emittance area the initial conditions

$$z = z_0 = 0 \ \Rightarrow \ y = y_0, x = x_0, \frac{\mathrm{d}y}{\mathrm{d}z} = 0, \frac{\mathrm{d}x}{\mathrm{d}z} = 0 \tag{18}$$

lead to unique solutions of the subsequent special forms

$$x_{p_0}(z) = x_0 \cos(\omega_0 z) \ , \ y_{p_0}(z) = y_0 \cos(\omega_0 z) \tag{19}$$
$$x'_{p_0}(z) = -x_0 \omega_0 \sin(\omega_0 z) \ , \ y'_{p_0}(z) = -y_0 \omega_0 \sin(\omega_0 z)$$

and represent ellipses parametrized by z (Fig.1a) characterized by the semi-axis x_0, ωx_0 and y_0, ωy_0 respectively. For further investigations, we present a detailed treatment of the emittance space $E_{xx'}$. It is proved below, if this calculation of emittance area within the beam guiding system is adequate yielding a statement about the constancy of emittance area and the goodness of the used approaches with reference to the measuring apparatus developed for this project (Fig 1e). This device for measuring the emittance consist mainly of a Faraday cup. The cup is movable in φ—direction and picks up the current in z-direction.

In case of varying the impulse p from p_0 to $p' = p_0 + \Delta p$ the new orbital equation using the same initial conditions (18)

$$x_{p_0+\Delta p}(z) = x_0 \cos(\omega_0 [1-\delta] z) \tag{20}$$
$$x'_{p_0+\Delta p}(z) = -x_0 \omega_0 (1-\delta) \sin(\omega_0 [1-\delta] z)$$

also describes an orbit of an ellipse with a changed angular velocity $\omega' = \omega_0(1-\delta)$ (Fig.1).

Defining $\delta = \Delta p / p_0$ as the ratio of the variation Δp and the impulse p_0 at the beginning, the subsequent emittance area (Fig.1d)

$$\Delta F_{xx'} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_0 & -x_0\omega_0^2 L & 0 \\ x_0 & -x_0\omega_0^2 L(1-\delta) & 0 \end{vmatrix} = 2|\delta|\, L\, x_0^2 \omega_0^2 \tag{21}$$

is evident and the above formula is valid for all p, $p_0 \leq p \leq (p_0 + \Delta p)$ and where $L = 2A$ has been introduced. The term $\omega^2 L$ equals f^{-1}, f beeing the focal length of the electrostatic lense. The increase of emittance area therefore reads

$$\Delta F_{xx'} = \frac{1}{f} 2|\delta| x_0^2 = |x_{0(max)}|\ 2\ |\delta| \tag{22}$$

where $x_{0(max)} = x_0$ and $x'_{(max)} = x_0\omega_0^2 L$ have been used. The difference of the angles

$$\Delta x' = x'_{p+\Delta p}(L) - x'_{p_0}(L) \tag{23}$$

can be expressed as

$$\Delta x' = x_0 \omega_0^2 L\ 2\delta \tag{24}$$

so that

$$\Delta x' = \frac{x_{E_1} - x_{E_2} - x_{a_1} + x_{a_2}}{a_2 - a_1} \tag{25}$$
$$x_E = r\ \cos\varphi_E \quad , \quad x_a = r\ \cos\varphi_a$$

follows considering the geometry of the measuring apparatus (Fig. 1e).

V. Conclusion

For definite assumptions of technical relevance, the Hamiltonian of electrostatic devices is separable. In that case a good approximation can be achieved with the assumption $z \gg R$ regarding.

The assumption $z \gg R$ is evident, for instance in the accelerator-technique of nuclear physics the component of focussing possesses a greater distance from the target chamber. This statement is demonstrated in eq.(16).

Eq.(25) represents a simple solution for construction of an apertural measurement of the emittance.

References

[1] H. Heydari, "Untersuchungen zum Einfluß der Zustandsfunktionen auf Wirkungsquerschnitte direkter p,γ-Reaktionen bei Energien im 100 keV-Bereich und Projektierung eines Kaskadenbeschleunigers zur Messung von p,γ-Einfangquerschnitten", Doctoral Thesis, TU Berlin, 1988

[2] D. Lennartz, "Phasenraummessungen und Untersuchung der Strahlparameter bei externer Energievariation an einem 20 MeV Protonenzyklotron" Doctoral Thesis, TU Berlin, 1977