Adiabatic Invariance for Spatially Dependent Accelerating Structures

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Abstract

The adiabatic dynamics of charged particles in accelerating structures is significantly altered if these structures vary in space rather than in time. Spatial variation occurs in, for example, free-electron lasers and radio-frequency quadrupoles. The adiabatic invariants for slow temporal and slow spatial variations differ. This causes the longitudinal emittance of adiabatically trapped beams to differ in the two cases. The number of particles trapped in each accelerating bucket also differs. We present analytic and numerical results to clarify these ideas.

I. INTRODUCTION

For a general Hamiltonian $H(q,p,\varepsilon t)$, which depends slowly on the time variable, the adiabatic invariant is the action:

$$I(E,\varepsilon t) \equiv \oint p|_{H=E}\, dq. \qquad (1)$$

The action is the phase-space area enclosed by a contour of H at fixed time t. *Slowly* means that the particle executes many oscillations before the parameters of the Hamiltonian change significantly (indicated formally by taking $\varepsilon<<1$). Invariance of the action allows one to solve for the energy at one time in terms of the energy at another time. Adiabatic theory has also been used to predict the phase-space area occupied by a beam adiabatically trapped in an accelerating potential.

We analyze the case of adiabatic, spatially varying, accelerating structures. We show that the usual adiabatic invariants for the temporally varying case are not invariant in the spatially varying case. This affects, in particular, the arguments that give the area occupied by a beam that has been adiabatically trapped in a *spatially* varying structure. Our results apply to problems such as beam trapping in the accelerating potential of an RFQ[1] or in the ponderomotive (decelerating) potential of free-electron lasers[2] and to the rf heating of a tokamak plasma.[3]

II. SLOW TEMPORAL VARIATION

The *wave Hamiltonian*,

$$H(q,p,\varepsilon t) = \frac{p^2}{2m} + e\Phi(\varepsilon t)\cos(k[q - u(\varepsilon t)t]). \qquad (2)$$

describes the longitudinal motion in an accelerating potential. Here, e and m are the mass and charge of the particle. The amplitude of the potential is Φ. The particles oscillate in a trough of phase velocity u and length $2\pi/k$.

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For trapped particles the adiabatic invariant is the phase-space area given by the usual closed-loop integral $\oint p\,dq$. However, for passing (or untrapped) particles, the loop integral is over one period (from $kq=0$ to $kq=2\pi$). We are, thus, able to define three different areas, or action functions, $I_\pm(E,\varepsilon t)$ and $I_T(E,\varepsilon t)$, which give the loop integrals along the contour of energy E at time t for particles passing above (+) or below (-) or trapped (T) inside the stable region.

This information determines the longitudinal emittance (phase-space area) of a beam that becomes trapped. We suppose for example in Fig. 1 that initially the amplitude of the wave vanishes, and the beam is at momentum $p_i=m(u+\Delta)$. At this time the value of the adiabatic invariant is the area under the curve $p=p_i$: $2\pi m(u+\Delta)/k$. As the amplitude grows (with u remaining constant), the contour of H for these particles distorts, but has below it always the same area. However, eventually there comes a time, called the crossing time t_x, when the area under the more positive part of the separatrix equals the initial value of the adiabatic invariant. Beyond this time there is no passing-particle curve having the same area. Thus, the particles must become trapped. Furthermore, these particles are on and remain on a trapped contour of H containing area $(16m/k)A_x^{1/2}$, where $A_x \equiv A(\varepsilon t_x)=e\Phi(\varepsilon t_x)/m$, since that is the area enclosed by the separatrix at the time of crossing.

These results determine the phase-space area occupied by an adiabatically trapped beam of particles. A beam extending from $p=mu$ to $p=m(u+\Delta)$ occupies an area per wavelength of $2\pi m\Delta/k$. As the wave amplitude increases from zero, first the particles with $p=u$ become trapped. As the amplitude grows, the larger momentum particles become trapped. The largest momentum particles become trapped at an amplitude A_x satisfying $2\pi\Delta=8A_x^{1/2}$. This result is obtained by equating the initial action with the value of the action on the separatrix.

III. ADIABATIC THEORY FOR SPATIAL VARIATION

For linear structures, such as RFQ's or free-electron lasers, the Hamiltonian (2) is not appropriate. The potential instead has an amplitude that varies spatially:

$$H(q,p,t) = \frac{p^2}{2m} + e\Phi(\varepsilon q)\cos(k[q - u(\varepsilon q)t]). \qquad (3)$$

Now the particles see an amplitude growing as they enter the accelerating structure. After being trapped in a stable bucket they may be accelerated or decelerated depending on how the phase velocity u changes with position.

To determine the new adiabatic invariant we introduce the phase-space variational principle, which states that the correct dynamics makes stationary the integral,

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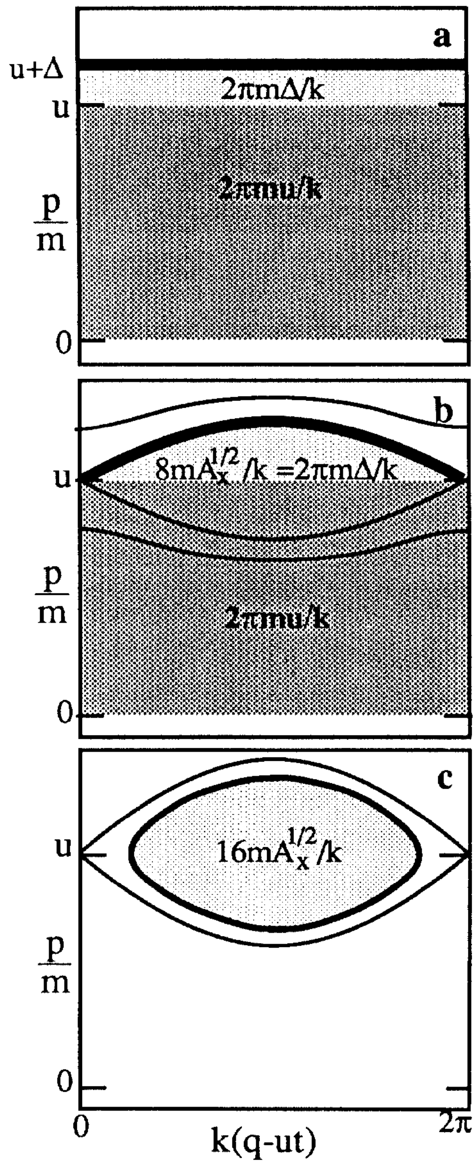

Fig. 1. Adiabatic trapping of a beam in time.

$$\mathcal{A} \equiv \int [p\,dq - H(q,p,\varepsilon t)\,dt], \tag{4}$$

of the phase-space differential action, $d\mathcal{A} \equiv p dq - H dt$. For historical reasons the quantity $\mathcal{A}$, which is a line integral along the trajectory, is also known as the action. The other action (1) is related to $\mathcal{A}$ by being the integral around a closed loop of constant H at constant t. The Euler-Lagrange equations for the functional (4) are exactly Hamilton's equations. Adiabatic theory follows when the Hamiltonian H is a slow function of the independent variable (t). In this case, the loop integral of the other conjugate pair (q,p) at constant slow pair (t,H) is the adiabatic invariant.

For the spatially varying case, $\mathcal{A}$ has the following form:

$$\mathcal{A}_q \equiv \int [p\,dq - H(q-ut,p,\varepsilon q)\,dt], \tag{5}$$

which is not the form required by adiabatic theory. However, we can put $\mathcal{A}_q$ in the appropriate form by first subtracting the total differential $mu^2dt/2$, then adding and subtracting the term (pudt) to obtain

$$\mathcal{A}_q \equiv \int [(p + K/u)\,d(q - ut) - (K/u)\,dq] \;, \tag{6}$$

where

$$K(q-ut,p,\varepsilon q) \equiv \frac{(p-mu)^2}{2m} + e\Phi(\varepsilon q)\cos(k(q-ut)) \tag{7}$$

is equal in value to the Hamiltonian obtained by transforming to the frame moving with the phase velocity.

The action integrand is now in the form needed to determine the spatial adiabatic invariant. The new momentum $z \equiv p+K/u$ is conjugate to the fast variable q-ut. The new Hamiltonian K depends slowly on the independent variable q. Thus, the new adiabatic invariant is given by the integral,

$$J = \int (p+K/u)\,d(q-ut) = \int p\,d(q-ut) + (K/u)\int d(q-ut) \;, \tag{8}$$

along a contour of constant K. The last equality in Eq. (8) follows from the fact that K is held constant in the integration.

For trapped particles, the phase (q-ut) begins and ends at the same point. Hence, the second term vanishes and we have,

$$J_T(I) = \oint p\,d(q-ut) \equiv 2I. \tag{9}$$

The last equality defines the variable I, which for trapped particles is simply half the enclosed area. This factor of 1/2 makes I a continuous phase-space variable.

For passing particles, the phase q-ut does not begin and end at the same value, but changes by $2\pi/k$. Thus, the adiabatic invariant for ± passing particles is given by

$$J_\pm(I,\varepsilon q) = I + 2\pi K(I,\varepsilon q)/ku \;, \tag{10a}$$

where

$$I(\varepsilon q,K) \equiv \int_0^{2\pi} (p-u)\,d(q-ut) \tag{10b}$$

is the integral along a contour of constant K. The convention chosen here of integrating in the direction of increasing phase q-ut makes I a signed quantity; it is positive (negative) for positive (negative) passing particles.

IV. Trapping in the Spatially Varying Case

Invariants of the motion permit understanding of the dynamics without solving the differential equations. Because the adiabatic invariant $J(I,\varepsilon q)$ (referring collectively to the three functions $J_\pm$ and J_T) is conserved, the trajectory stays on contours of this function. This allows us to determine the value of I as the particle moves through the accelerating structure.

We show in Fig. 2 a contour plot of $J(I,\varepsilon q)$, with contours divided into four regions by the I axis and the heavy curves outlining what appear to be lips. This outline indicates the location of the separatrix, which is given by $I = \pm 8(m/k)A^{1/2}$. The lips vanish at large distances, where the accelerating potential vanishes. Above the lips are the positive passing particles, and below the lips are the negative passing particles.

The trapped particles are in the upper lip, because for trapped particles I is positive according to Eq. (9). The lower lip is an unphysical region.

We first discuss the trajectories in Fig. 2 which do not intersect the lips (i.e. do not become trapped). The top orbit, labeled (a), is an orbit that passes right over the structure. Similarly, the orbit labeled (h) is a left moving orbit that passes over the structure in the opposite direction. The orbit (g) and its mirror reflection on the right begin moving into the potential, but then are reflected. The corresponding particles are reflected by the ponderomotive potential[4] associated with the oscillating electric field. The orbit (f) moves into the accelerating structure, but its velocity is sufficiently negative of the phase velocity that it does not become trapped by the potential.

Now we turn to the trajectories that do become trapped. The trajectory (b) collides with the separatrix, at which point the corresponding particle becomes trapped. For trapped particles, I is the adiabatic invariant, and so I remains constant. Eventually this orbit exits the structure at the other side, becoming untrapped. As indicated by the arrows, this orbit may exit onto a positive passing trajectory (c) or a negative passing trajectory (e) connected by the downward arrow. That is, the trajectory can exit out the top half of the phase-space separatrix or the bottom half. This phenomenon, *beam splitting*, implies that a single beam becomes two beams after trapping and detrapping take place. The time reversed phenomenon is that there are two trajectories, (b) and (d), which trap at the same action.

The trapping areas follow from application of this adiabatic theory. We consider a beam of particles with momentum $p_i=m(u+\Delta)$ far from the accelerating structure. Such particles are in the positive passing state and have an initial value of the adiabatic invariant given by,

$$J_{+i} = (2\pi m/k)\left(\Delta + \tfrac{1}{2}\Delta^2/u\right). \qquad (11)$$

These particles trap when the value of the positive passing adiabatic invariant on the separatrix,

$$J_{+sx} = 8mA^{1/2}/k + 2\pi mA/ku, \qquad (12)$$

equals the initial value of the adiabatic invariant. Equating (11) and (12) shows that these particles become trapped at an amplitude A_x satisfying

$$\Delta + \tfrac{1}{2}\Delta^2/u = 4\,A_x^{1/2}/\pi + A_x/u, \qquad (13)$$

which implies that these particles are spread throughout an area

$$16mA_x^{1/2}/k = (32mu/k\pi)\left(\sqrt{1+\pi^2\Delta/4u+(\pi\Delta/2u)^2/2} - 1\right). \qquad (14)$$

For small values of the momentum difference Δ (and, therefore, trapping amplitude), (14) reduces to the result obtained in the time varying case.

V. CONCLUSIONS

The case of adiabatic, spatially varying accelerating structures differs significantly from that of temporally varying adiabatic structures. The adiabatic invariant for passing particles is not the action (10b), but rather the "new" adiabatic invariant (10a). The phase space area finally occupied by a trapped beam in the temporally varying case is invalid; that result is replaced by (14). In addition, ponderomotive reflection appears.

VI. REFERENCES

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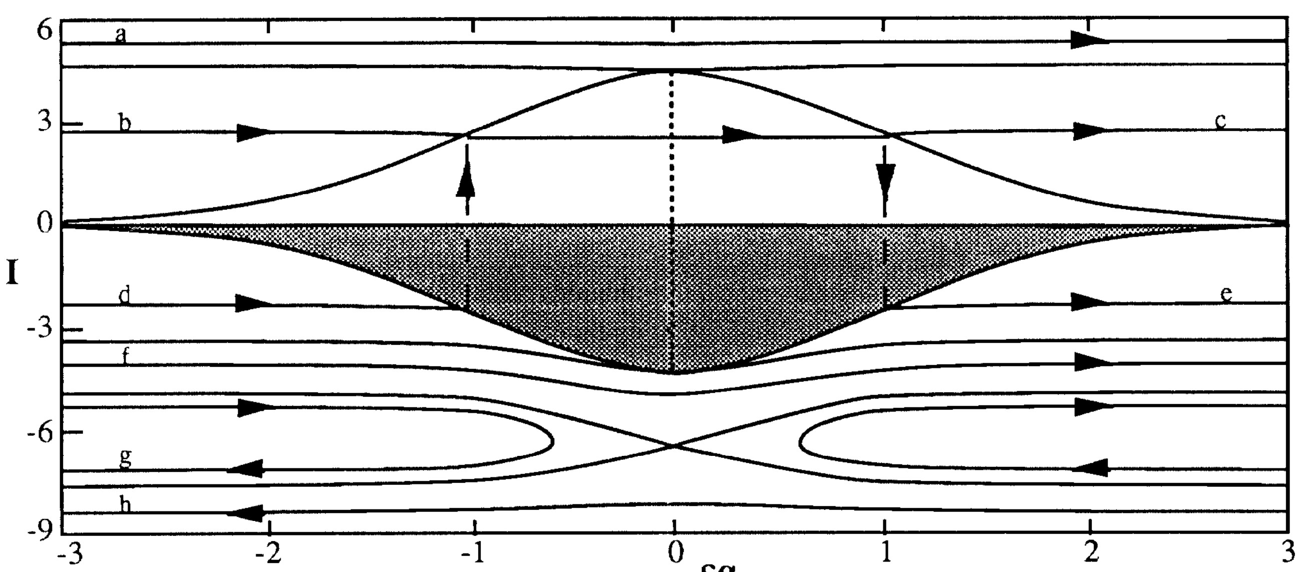

Fig. 2. Contours of the spatial adiabatic invariant for Φ gaussian in shape and $ke\Phi_{max}=0.4mu^2$.