Further Dynamic Aperture Studies on a Wiggler-Based Ultra-Low-Emittance Damping Ring Lattice *

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Abstract

Further dynamic aperture studies on an ultra-low emittance damping ring lattice are presented. A past conference paper[1] explained how the fast damping rate, the low emittance and the large dynamic aperture are acheived for this lattice. Dynamic aperture improvement with octupole correction was also reported. In this paper the dynamic aperture improvement is emphasized with a more systematic derivation and study of the octupole correction. Also, the modified sextupole proposal of Cornacchia and Halbach[2] is applied to the damping ring lattice.

I. INTRODUCTION

The ultra-low emittance damping ring lattice reported in a past conference paper[1] makes use of long dispersion-free straight sections filled with strong wigglers to produce fast synchrotron radiation damping. The lattice also has large radius arcs with strongly-focusing FODO cells to produce low quantum excitation. As Wiedemann points out in a proposal to lower the emittance of PEP[3], both features yield a very low equilibrium emittance. In the damping ring lattice proposed in [1], an emittance of 2.5×10^{-11} m-rad for a beam energy of 4 GeV is acheived ($\epsilon_n = \epsilon\gamma = 2 \times 10^{-7}$ m-rad).

To maximize the dynamic aperture (the range of stability for transverse oscillations), FODO cell achromats as defined by Brown and Servranckx[4] with *non-interleaved* sextupoles are adopted. Although an interleaved achromat arrangement can accomodate a larger number of sextupoles, thus reducing the individual strengths, the dynamic aperture suffers greatly. In both interleaved and non-interleaved cases the second-order geometric aberration sextupole terms are made to vanish. Thus, vanishing second-order aberration terms does not guarantee the maximum possible dynamic aperture[5]. In general, sextupoles interact with each other to produce higher-order aberration terms which become important for low emittance lattices. Non-interleaved achromats, as are implemented in the damping ring proposed above, are simply a way to prevent sextupoles from interacting in this way. Further examination shows that the main geometric aberrations produced in these optics modules are due to the lengths of the sextupoles.

In this paper, new analytical formulae for the sextupole length aberrations are derived. Octupoles can be inserted into the lattice to selectively cancel some of these aberration terms, thus enlarging different parts of the dynamic aperture. Numerical tracking of particle trajectories confirms this.

II. SEXTUPOLE LENGTH ABERRATION

The equations of transverse motion inside a sextupole are

$$x'' = -\frac{1}{2}m(x^2 - y^2), \quad (1)$$

$$y'' = mxy. \quad (2)$$

where m is the normalized sextupole strength, $(e/cp)\partial^2 B_y/\partial x^2$. These equations will be integrated along the longitudinal coordinate s using an iterative method. One starts with constant initial solutions $x(s) = x_0$, $x'(s) = x_0'$, $y(s) = y_0$, $y'(s) = y_0'$,

and iterate the following 4 steps,

$$x'(s) \leftarrow x_0' + \int_0^s \left(-\frac{1}{2}m(x(s)^2 - y(s)^2)\right) ds', \quad (3)$$

$$x(s) \leftarrow x_0 + \int_0^s x'(s') \, ds', \quad (4)$$

$$y'(s) \leftarrow y_0' + \int_0^s (mx(s)y(s)) \, ds', \quad (5)$$

$$y(s) \leftarrow y_0 + \int_0^s y'(s') \, ds'. \quad (6)$$

until the resulting functions (polynomials in s) are of sufficient accuracy. Keeping only terms cubic in coordinate variables, the sextupole exit coordinates take the form

$$x_i(l) = \sum_{j=1}^{4} R_{ij}(l)x_{j0} + \sum_{j=1}^{4}\sum_{k=1}^{j} T_{ijk}(l)x_{j0}x_{k0}$$
$$+ \sum_{j=1}^{4}\sum_{k=1}^{j}\sum_{l=1}^{k} U_{ijkl}(l)x_{j0}x_{k0}x_{l0} \quad (7)$$

for $1 < i < 4$ where $x_1 \equiv x$, $x_2 \equiv x'$, $x_3 \equiv y$, and $x_4 \equiv y'$, and l is the sextupole length. The notation and formalism of nonlinear matrix elements is that of K.Brown[4]. The R_{ij} matrix elements represent a drift space of length equal to the sextupole length. The T_{ijk} matrix elements are proportional to the integrated sextupole strength. An achromat is designed so that the T_{ijk} terms for $1 < i, j, k < 4$ contributed by the sextupole pair within the achromat cancel exactly. The U terms are called third-order matrix elements, and are the most important terms for an achromat with long sextupoles, since the contributions of each sextupole of the pair add together. It is therefore sufficient to examine the U terms of one sextupole for local octupole compensation.

A. Largest Third-Order Matrix Elements

Out of 80 possible U_{ijkl} terms (for $1 < i, j, k < 4$), 40 are non-zero. Most of these are small. The strongest terms are found by converting the $x_i x_j x_k$ factors into normalized coordinates. The normalized coordinates are $u = x/\sqrt{\beta_x}$ and $\dot{u} = \sqrt{\beta_x}x' + \alpha_x x'/\sqrt{\beta_x}$ for the horizontal plane, and $v = y/\sqrt{\beta_y}$ and $\dot{v} = \sqrt{\beta_y}y' + \alpha_y y/\sqrt{\beta_y}$ for the vertical plane. In a linear lattice, the particle trajectories in normalized coordinate phase space are circles (i.e., $u^2 + \dot{u}^2 = 2J$ is an invariant). Inserting the normalized coordinate definitions into equation (7) gives cubic terms in u, $\dot{u}$, v, and $\dot{v}$, which perturb the value of the linear invariant at every sextupole location. The coefficients of the cubic terms are the U_{ijkl} times some power of $1/\beta_{x,y}$ where l is the sextupole length. Normally in a storage ring, $l \ll \beta_{x,y}$. Therefore the most important nonlinear matrix elements have the lowest power of $1/\beta_{x,y}$.

One can repeat the same integration procedure above for particle motion in an octupole field:

$$x'' = -\frac{1}{6}O(x^3 - 3xy^2), \quad (8)$$

$$y'' = -\frac{1}{6}O(y^3 - 3x^2y), \quad (9)$$

where $O = (e/cp)\partial^3 B_y/\partial x^3$. The largest third-order matrix elements for octupoles are the same type as those of sextupoles.

If both sextupole and octupole fields are combined in a magnet, the largest third-order aberration terms are

$$\Delta \dot{u} = \left(-\frac{1}{6}Ol + \frac{1}{12}m^2l^3\right)\beta_x^2 u^3 + \left(\frac{1}{2}Ol + \frac{1}{12}m^2l^3\right)\beta_x\beta_y uv^2,$$
(10)

$$\Delta \dot{v} = \left(-\frac{1}{6}Ol + \frac{1}{12}m^2l^3\right)\beta_y^2 v^3 + \left(\frac{1}{2}Ol + \frac{1}{12}m^2l^3\right)\beta_x\beta_y u^2 v.$$
(11)

These two important equations summarize the largest aberration terms for a long sextupole and connect the aberrations from sextupoles with those of octupoles. They are the basis of octupole correction.

III. LOCAL OCTUPOLE CORRECTION

An octupole field with strength O can be superimposed on the sextupole field to cancel one or the other higher-order aberration terms, but not both, unfortunately. However, one of the two terms in each of these equations is much greater than the other because the sextupoles are placed where $\beta_x > \beta_y$ or $\beta_y > \beta_x$ for effective chromaticity correction. The larger of the two terms must obviously be targeted for cancellation using the octupole field strength as an adjustable parameter. The appropriate octupole integrated strength is

$$Ol = \frac{1}{2}m^2l^3.$$
(12)

For instance, if $\beta_x > \beta_y$, this choice of octupole strength cancels the u^3 term in equation (10) and also the small v^3 term in equation (11), but increases the value of the coefficient of the coupling terms uv^2 and u^2v by a factor of four. Equations (10) and (11) become

$$\Delta \dot{u} = \frac{1}{3}m^2l^3\beta_x\beta_y uv^2$$
(13)

$$\Delta \dot{v} = \frac{1}{3}m^2l^3\beta_x\beta_y u^2 v$$
(14)

For oscillations of equal invariant value in both planes (i.e. $v \approx u$) the ratio of the coupling term to the u^3 term is $4\beta_y/\beta_x$, which is of order one for FODO lattices. The coupling effect on the dynamic aperture from tracking doesn't seem to increase that much as a result.

For the other case, $\beta_y > \beta_x$, the largest term is the v^3 term, which vanishes when the same octupole strength $O = -\frac{1}{2}m^2l^2$ is used. Note that the sign of the octupoles are the same in both cases. The reason is that the main nonlinear effect produced by the interaction of two sextupoles must always scale with the product of the sextupole strengths. In the case of sextupole self-interaction, the strength is squared, and the sign disappears.

A. Dynamic Aperture with Octupolar Correction

The octupole correction described above will be tested by tracking the damping ring lattice with an octupole family correcting each sextupole family. The damping ring uses two sextupole families. To simplify the discussion the sextupoles that correct the horizontal chromaticity are called SF sextupoles and the SF-correcting octupoles, OF octupoles. Similarly, the sextupoles that correct the vertical chromaticity are called SD and the correcting octupoles, OD octupoles. Figure 1 shows schematically where the SF sextupoles are placed relative to the focusing quadrupoles within the 90° FODO cell achromat of the damping ring lattice. The required integrated strengths of the octupoles are listed in Table 1. Ideally the octupolar field should be superimposed on the sextupole field for proper cancellation of aberration terms. In the tracking the octupoles were modeled as a thin-lens inserted in the center of each sextupole. Also, the sextupoles in the tracking were each modeled as two kicks spaced $l/2$ apart. In this approximation the third-order aberration term of the sextupole is reduced by a factor of 3/4. The correcting octupole strengths are adjusted to cancel the third-order aberration terms of the two-kick sextupole.
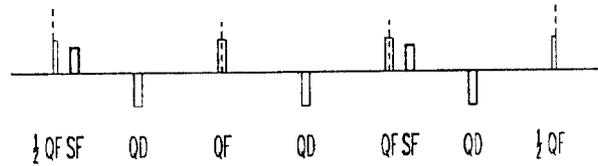
Figure 1
Achromat module for SF sextupoles

Table 1
Required octupole strength for correction

| Octupole | Ol (m^{-3}) | Associated sextupole | |
		ml (m^{-2})	l (m)
OF	41.3	-16.6	0.3
OD	43.8	17.1	0.3

The difference in strength between the two-kick model and the uniform model should not change the conclusions.

One can look at the octupole correction in stages. First the octupoles (OF) that correct the SF's where $\beta_x \gg \beta_y$ are inserted. The calculated dynamic aperture for $\beta_x=12.2$ m and $\beta_y=2.2$ m is shown in Figure 2. A large improvement is observed in the stability in the horizontal plane, as one would expect from equation (10), and none in the vertical plane since the cancelled v^3 term in equation (11) is very small. The uncorrected SD's are responsible for the vertical aperture limit. The OF octupole also has the effect of correcting the horizontal tune shift with amplitude. When the octupoles (OD) that correct the SD's are inserted where $\beta_y \gg \beta_x$ (see Figure 3) the dynamic aperture in the vertical direction is improved.
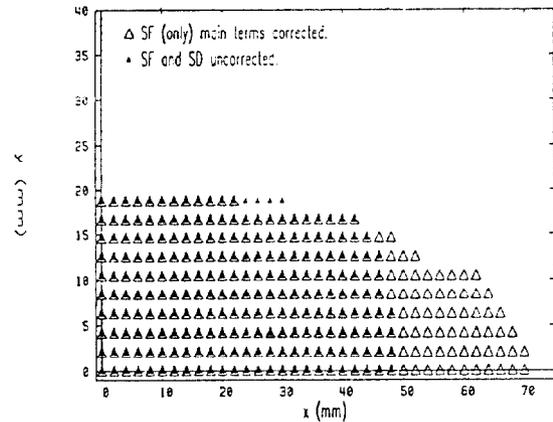
Figure 2
Dynamic aperture with SF-correcting octupole.

When both octupole families are turned on (Fig. 4), the stability along both axes is improved compared to the case with no octupoles. However, the dynamic aperture along both axes is smaller than the best possible values achieved when only one octupole family is used. There is no improvement in the area of the x-y plane where both x and y are large. This is understandable because the octupoles do not remove nonlinear coupling, but in fact increase it.

The dynamic aperture can be improved (up to 40%) and shaped with octupoles according to one's need. For instance if only horizontal aperture is required, then only the octupoles associated with the SF sextupoles should be turned on.

As an aside, a straightforward way of reshaping the dynamic aperture would be to redistribute the number of SF modules
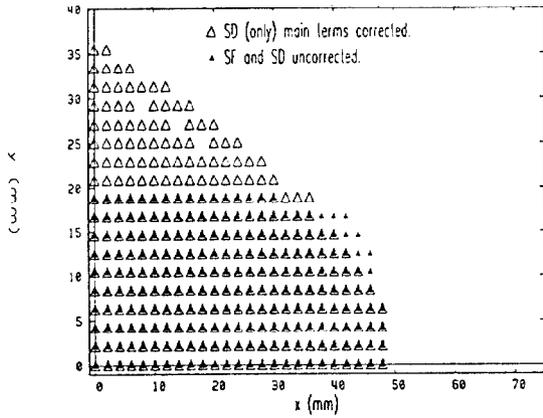
Figure 3

Dynamic aperture with SD-correcting octupole.
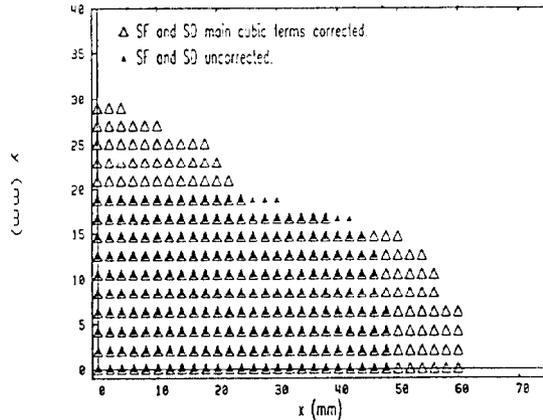
Figure 4

Dynamic aperture with SF- and SD-correcting octupole.

and SD modules. If for some reason the horizontal dynamic acceptance needs to be much greater than the vertical one, then the achromats used for SD sextupoles can be converted over to SF sextupoles. Repartitioning the modules in this way would reduce the individual SF sextupole strength and increase the individual SD sextupole strength. The horizontal dynamic aperture will then increase at the expense of the vertical dynamic aperture.

IV. APPLICATION OF MODIFIED SEXTUPOLES

After the above work with octupole correction was completed, a report on the design of modified sextupole magnets for dynamic aperture improvement in storage rings was published by Cornacchia and Halbach[2]. They propose reshaping the poles of sextupoles in order to reduce the magnetic field along a chosen transverse direction, while preserving the sextupole field symmetry near the axis. Outlying particles of a stored gaussian beam would feel a weaker nonlinear field and follow a stabler trajectory, while the core of the beam feels the regular sextupole field. This has the effect of increasing the dynamic aperture. While the chromaticity of the core of the beam is corrected, the chromaticity of large-amplitude orbits is not. Fortunately, the head-tail instability, for which the chromaticity correction is needed, is a collective effect, and it's not necessary for all of the beam to have zero or positive chromaticity.

Cornacchia and Halbach mention many possible field distributions for modified sextupoles. A modified sextupole field design that is probably the easiest to implement is the following[2]:

$$B^* = B_x - iB_y = -iAz^2 \exp(\kappa z^2) \qquad (15)$$

where $z = x + iy$, A is a constant, and κ is a decay parameter

to be adjusted. Because of the $\exp(\kappa z^2)$ factor this sextupole is called a gaussian sextupole.

If $\kappa = 0$ we have an ordinary sextupole field. For $\kappa < 0$, the sextupole field eventually decays along the horizontal axis, but grows indefinitely along the the vertical axis, while the reverse is true for $\kappa > 0$. It would seem that the dynamic aperture would increase in one plane at the expense of the other. However, β_x and β_y are normally very different at sextupoles. One finds that the SF family of sextupoles (where $\beta_x > \beta_y$) should have $\kappa < 0$, and that the SD family of sextupoles (where $\beta_y > \beta_x$) should have $\kappa > 0$. Since the gaussian sextupole is very non-linear and not a simple multipole, its study using matrix elements is difficult. It is therefore difficult to predict which absolute value of κ will improve the dynamic aperture the most for a given lattice. A rough estimate would be to set κ such that $\kappa z^2 \approx$ -1 at the dynamic aperture limits measured at the sextupole position.

I optimized by trial and error the dynamic aperture for the damping ring, treating κ for each of the sextupole families as independent adjustable parameters. The dynamic aperture for the optimal gaussian sextupoles with κ_{SF} = -75 m^{-2} and κ_{SD} = 250 m^{-2} is shown in Figure 5. Again, the dynamic aperture
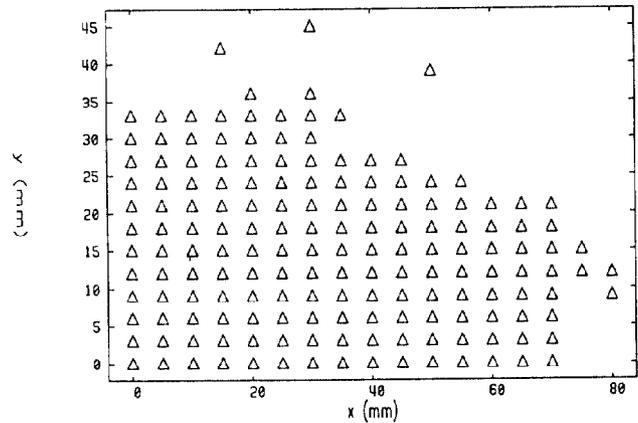
Figure 5

Dynamic aperture with gaussian sextupole.

is limited by sextupole length effects. In this case however, the dynamic aperture improvement along both axes (about 40%) is better than that of octupole correction.

V. CONCLUSION

The large dynamic aperture of non-interleaved achromat-based lattices can be improved further with correcting octupoles. The modified gaussian sextupole was also successfully applied to the low-emittance damping ring, and may be worth studying further.

VI. REFERENCES

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1635