Bunched Beam Longitudinal Stability

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Abstract

Instabilities driven by narrow-band impedances can be stabilized by Landau damping arising from the synchrotron frequency spread due to the nonlinearity of the rf waveform. We calculate stability diagrams for various phase space distributions. We find that distributions without tails are unstable in the 'negative mass' regime (inductive impedance below transition or capacitive impedance above transition). We also find that longitudinal instability thresholds of the (usually neglected) higher order radial modes are lower than expected. For example, the next to lowest dipole mode has a lower threshold than the lowest sextupole mode even though the latter has the larger growth rate in the absence of Landau damping.

I. INTRODUCTION

It is difficult to calculate coupled-bunch instabilities for the case of arbitrary impedance functions. Also, it is usually not very illuminating because little insight is gained and strategies for stabilization are not easily deduced. In this paper, we find thresholds and growth rates arising from parasitic narrow-band resonances. We closely follow the formalism developed by Balbekov [1], but use a different longitudinal phase space distribution function. In particular, we use a distribution which, as a function of synchrotron amplitude, is parabolic near the centre. Such a shape agrees with measured bunch profiles and is moreover expected from thermodynamic considerations.

II. THRESHOLD

Lebedev [2] showed that the Vlasov equation for longitudinal phase space can be cast into the form of an eigenvalue problem for the beam current perturbation harmonics. Assume there is a parasitic resonator which is narrow-band in the sense that the quality factor is large compared with the ratio of resonator to beam frequency ($=n/h$ where h is the harmonic number). In that case, the impedance can couple to a coupled-bunch mode at only one frequency so the eigenvalue matrix reduces to 1×1. The dispersion equation for the harmonic n ($\omega = n\omega_0+\Omega$, where ω_0 is the revolution frequency) is given by

$$\frac{Z_0}{Z_n(\Omega)} = Y_n(\Omega) \equiv -2i\omega_{s0}\sum_{m=-\infty}^{\infty} m\int_0^{\mathcal{E}_0}\frac{F'(\mathcal{E})|I_{mn}(\mathcal{E})|^2}{\Omega - m\omega_s(\mathcal{E})}d\mathcal{E} \tag{1}$$

where

$$I_{mn}(\mathcal{E}) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\left[im\psi + i\frac{n}{h}\phi(\mathcal{E},\psi)\right]d\psi \tag{2}$$

and

$$Z_0 = n\frac{\gamma\beta^2\eta E_0}{eI}\left(\frac{\Delta p}{p}\right)^2 = \frac{n}{h}\mathcal{E}_0\frac{V\cos\phi_s}{I}. \tag{3}$$

($\Delta p/p$ is the half-width of the momentum spread at base, I is the average current, V is the rf voltage.) $\omega_s(\mathcal{E})$, ψ and $\mathcal{E}$ are the frequency, angle and action of synchrotron oscillations, $\mathcal{E}_0$ is the maximum action in the bunch: for linear oscillations, the rf phase of a particle w.r.t. the synchronous phase is $\phi=\sqrt{\mathcal{E}}\cos\psi$. $F(\mathcal{E})$ is the distribution function of a bunch; it is normalized according to $\int(F(\mathcal{E})/\omega_s(\mathcal{E}))d\mathcal{E} = \mathcal{E}_0/\omega_{s0}$.

The threshold curve $Y_n^{\rm th}(\Omega)$ can be found from (1) by letting the imaginary part of Ω approach zero. We consider only cases of weak nonlinearity, where the function $I_{mn}(\mathcal{E})$ (2) can be replaced by the Bessel function $J_m(\frac{n}{h}\sqrt{\mathcal{E}})$ because $\phi\approx\sqrt{\mathcal{E}}\cos\psi$. This restricts us to cases where the tune spread is less than around 20%.

It is clear from (1) that distribution functions F with large slopes are less stable. We consider bunches populated according to the density function

$$F(\mathcal{E}) = K(1-\mathcal{E}/\mathcal{E}_0)^\mu \tag{4}$$

($K\approx\mu+1$ for small synchrotron frequency spread). For $\mu<1$, F' is infinite at the beam edge so the threshold impedance is zero. This is illustrated in Fig. 2 where we have plotted stability diagrams in the impedance plane for the elliptic distribution (parabolic line density) μ=0.5, the parabolic distribution (parabolic in ϕ, linear in $\mathcal{E}$) μ=1, and μ=1.5. These are for $\sqrt{\mathcal{E}_0}$=95° in the μ=0.5 case, and the other cases had $\mathcal{E}_0$ scaled to maintain the same peak line density (see Fig. 1). The two diagrams are for resonators at n/h=2 and n/h=0.5 [1]. It can be seen that indeed for μ=0.5, the stability boundary passes through the origin. It also does so for μ=1 and the reason is more subtle: since F' remains finite at the beam edge, the integral in (1) diverges when Ω is exactly $m\omega_s(\mathcal{E}_0)$ because in that case only one side of the singularity is integrated over.

The stability boundaries in Fig. 2 are for the dipole mode (m=1). The other modes lead to additional loops in the impedance plane. An example is shown in Fig. 3 where up to m=4 is shown for the case of $\sqrt{\mathcal{E}_0}$=55° with distribution μ=1.5 for a parasitic resonance at n/h=5. Other such curves are summarized in Fig. 4 where reciprocal threshold shunt impedances are plotted as a function of frequency.

[1] No significance should be attached to the fact that in general we use round numbers for n/h. In fact, if n/h is exactly an integer or half integer (always the case if h is 1 or 2), the present stability analysis is not correct because frequencies (of either sign) separated by the rf frequency contribute to the same coupled-bunch mode. Effectively, this means that the present analysis ignores Robinson stability.

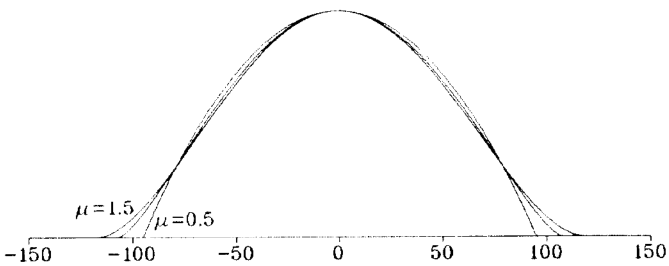

Figure 1: Line density profiles for the three distributions $\mu = 0.5$, 1.0 and 1.5 (see Eqn.4). With the x-axis in degrees, we get the stability diagrams below.

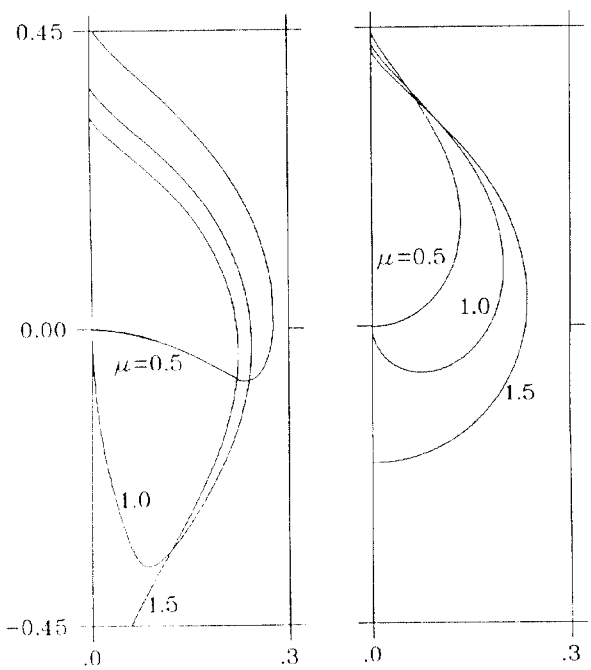

Figure 2: Stability boundaries in the impedance plane for the dipole mode for the three line densities of Fig. 1. The left plot is for n/h=2 and the right plot is for n/h=0.5. The impedance is in units of $V\cos\phi_s/I$. Scaled in this way, the diagram depends only upon the frequency (n/h) and the bunch length.

III. Growth Rate

An upper limit on growth rate $(1/\tau_m)$ of azimuthal mode number m is found from the m^{th} term in (1) by ignoring the synchrotron frequency spread and replacing Ω by $m\omega_s + i/\tau$. We get

$$\frac{1}{\tau_m} = -\frac{2m\omega_s}{n/h}\frac{R_{\text{sh}}I}{\mathcal{E}_0 V\cos\phi_s}\int_0^{\mathcal{E}_0} F'(\mathcal{E})J_m^2(n/h\sqrt{\mathcal{E}})d\mathcal{E}. \quad (5)$$

For the distribution family (4), Satoh [3] has solved this integral as an infinite sum and he has shown moreover that the individual terms in the infinite sum correspond to the different radial modes belonging to the azimuthal mode m:

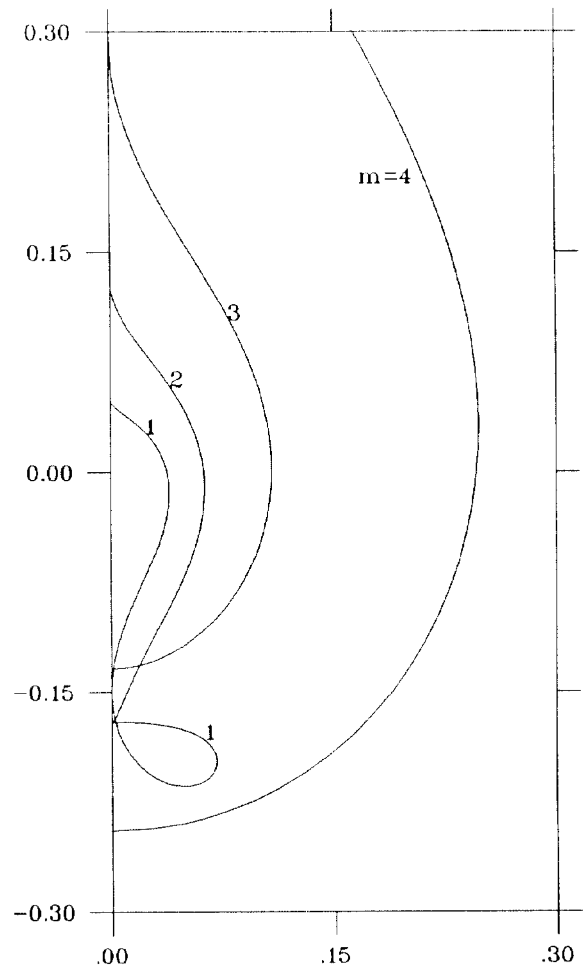

Figure 3: Stability diagram in the impedance plane for a parasitic near 5 times the rf frequency, $\sqrt{\mathcal{E}_0}$=55°. Notice that the dipole mode is peculiar in that it has an extra loop. This arises because there is more than one oscillation of the square of $J_1(5\sqrt{\mathcal{E}})$ for $\mathcal{E} < \mathcal{E}_0$, so there are two dipole modes; ordinary and extraordinary. Inside the extra loop, the extraordinary mode is stable and the ordinary dipole mode is unstable. Both of these dipole modes are linear combinations of the radial modes (k) discussed below.

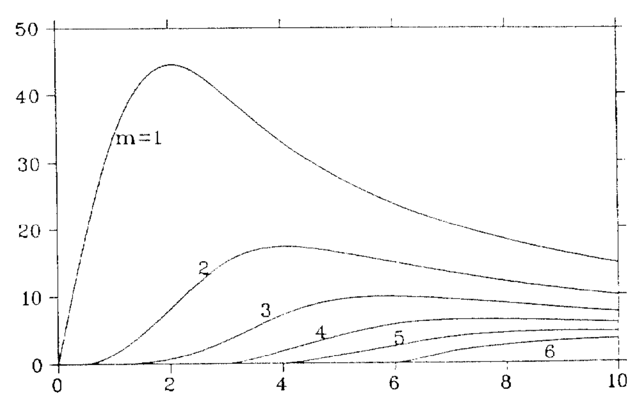

Figure 4: Reciprocal of the threshold R_{sh} (in units of $I/(V\cos\phi_s)$) vs. n/h for $\sqrt{\mathcal{E}_0}$=55° with distribution μ=1.5. Thresholds up to the dodecapole are shown.

$$\frac{1}{\tau_m} = \sum_{k=0}^{\infty} \frac{1}{\tau_{mk}} = m\omega_s \frac{n}{2h} \frac{R_{\rm sh} I}{V \cos\phi_s} \mu(\mu+1) \left(\frac{2}{X}\right)^{2\mu+2} \times$$
$$\times \sum_{k=0}^{\infty} \frac{(m+2k+\mu)\Gamma(k+\mu)\Gamma(m+k+\mu)}{k!(m+k)!} J^2_{m+2k+\mu}(X) \tag{6}$$

(X is an abbreviation for $\frac{n}{h}\sqrt{\mathcal{E}_0}$).

Both the individual terms of (6) and their sums have been plotted in Fig. 5. Take particular note of the rigid (k=0) mode for each m. These correspond to the form factors given by Sacherer [4] (Sacherer picture). The bottom plot of Fig. 5 is a summary plot of the sums $1/\tau_m$. This can be thought of as the Laclare [5] picture. The Sacherer picture is relevant for calculating growth rates from resistive impedances when a short-range wake like space charge lifts the degeneracy of radial modes and dominates in determining the thresholds. The Laclare picture is relevant when the short-range wake effects are small compared with those due to the narrow-band resonator.

IV. Discussion

A resonator can be represented by a circle tangent to the imaginary axis in the impedance plane. If no other impedance is present, the circle is centred on the real axis and it is clear (Fig. 2) that distributions with $\mu<1$ are unstable for any finite $R_{\rm sh}$. If there is also a negative imaginary impedance (eg. due to inductive wall effect below transition or space charge above transition) then distributions with $\mu \leq 1$ are unstable. In other words, tails are essential in this case. On the other hand, with capacitive impedance below transition, bunches can have line densities which are close to parabolic. This is consistent with the measured bunch shapes of the CERN PS Booster [4].

It is interesting to compare Figs. 4 and 5. Counter to intuition, it is not the mode with the fastest growth rate (in the absence of Landau damping) which has the lowest threshold. In fact, thresholds at any frequency go monotonically with azimuthal mode number and the dipole mode always has the lowest threshold. For example, at n/h=5 and $V\cos\phi_s/(IR_{\rm sh})$=23 in Fig. 4 only the dipole mode is unstable. It is clear that in this case it is predominantly the k=1 radial mode (see Fig. 5) that is being excited even though its growth rate is relatively small.

V. References

[1] V.I. Balbekov and S.V. Ivanov, Atomnaya Energiya, **60**, p.45 (1986).

[2] A.N. Lebedev, Atomnaya Energiya, **25**, p.100 (1968), and *Proc. VI Int. Conf. on High Energy Acc.*, p.284 (1967).

[3] K. Satoh, PEP-Note 357, May 1981.

[4] F.J. Sacherer, IEEE Trans. Nucl. Sci., **NS-20**, p.825 (1973), and IEEE Trans. Nucl. Sci., **NS-24**, p.1393 and p.1396 (1977).

[5] J.L. Laclare, CAS Proc., CERN 87-03, p.264.

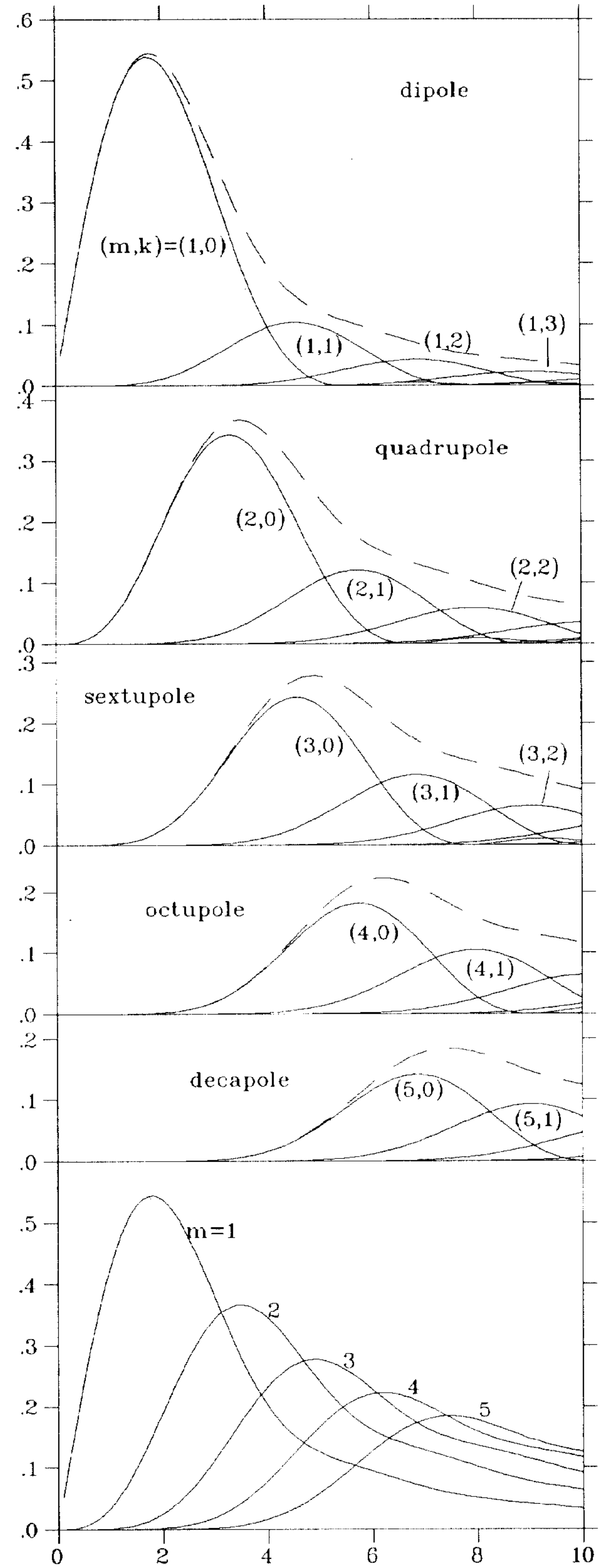

Figure 5: Growth rates (in units of $\omega_s R_{\rm sh} I/(V\cos\phi_s)$) vs. n/h for the modes indicated. The dashed curves and the curves in the summary plot at the bottom are the sums over k for any given m: they apply to the case of narrow-band impedance only, i.e. no splitting due to space charge.