Refinement of the Hamilton-Jacobi Solution using a Second Canonical Transformation*

W. E. Gabella†
Department of Physics,
University of Colorado,
Boulder, CO 84309

R. D. Ruth and R. L. Warnock
Stanford Linear Accelerator Center,
Stanford University,
Stanford, CA 94309

Abstract

Two canonical transformations are implemented to find approximate invariant surfaces for a nonlinear, time-periodic Hamiltonian. The first transformation is found from the non-perturbative, iterative solution of the Hamilton-Jacobi equation. The residual angle dependence remaining after performing the transformation is mostly eliminated by a second, perturbative transformation. This refinement can improve the accuracy, or the speed, of the invariant surface calculation. The motion of a single particle in one transverse dimension is studied in a storage ring example where strong sextupole magnets are the source of the nonlinearity. The refined transformation to action-angle variables, and the corresponding invariant surface, can attain accuracy similar to that of a good non-perturbative transformation in half the computation time.

I. The First, Non-Perturbative Canonical Transformation

We assume that we have solved the Hamilton-Jacobi equation iteratively to find an approximate invariant torus and the canonical transformation to an approximate set of action-angle variables for the full, nonlinear Hamiltonian[1] . In this paper, we discuss a refinement of this solution.

We start with the time-periodic Hamiltonian that describes the transverse motion of a single charged particle in a storage ring with sextupoles,

$$H(\Phi_0, I_0, s) = \frac{I_0}{\beta(s)} + \frac{1}{3!} S(s) (2\beta I_0)^{3/2} \cos^3 \Phi_0 \quad , \quad (1)$$

where $S(s)$ gives the strength in m^{-3} and distribution of the sextupoles around the ring. We have used the action-angle variables of the linear part of the Hamiltonian. The arclength around the storage ring $s \in [0, C]$ serves as the 'time' variable in the problem. In principle, the method for finding the non-perturbative solution is applicable to an arbitrary nonlinearity, as long as it is time-periodic. For single particle motion in storage rings, the dominant nonlinearity often comes from sextupole magnets.

The canonical transformation from the variables (Φ_0, I_0) to (Φ_1, I_1) is generated by $F_2(\Phi_0, I_1, s) = \Phi_0 I_1 + G^1(\Phi_0, I_1, s)$. The three different sets of action-angle variables will be discriminated by subscripts. For the two different generators G we use superscripts. The non-perturbative, numerical solution of the Hamilton-Jacobi equation described in Reference [1] yields the Fourier coefficients of G^1 on a finite mode set. The Fourier series is

$$G^1(\Phi_0, I_1, s) = \sum_{m \in M_0} g^1(m, I_1, s)\, e^{im\Phi_0} \quad . \quad (2)$$

The summation is over a finite set of modes indicated by M_0.

The numerical solution for G^1 serves as a perfectly good canonical transformation even if it does not yield the action-angle variables for the problem. It is a well-defined function of the variables (Φ_0, I_1, s), and it is very accurately periodic in s. We can implement the canonical transformation, allowing for a possible residual angle dependence, finding the Hamiltonian in the new variables:

$$H_1(\Phi_1, I_1, s) = \frac{I_1}{\beta(s)} + A(I_1, s) + V_1(\Phi_1, I_1, s) \quad . \quad (3)$$

The V_1 is defined such that its average over the new angle is zero. If G^1 were a perfect solution of the Hamilton-Jacobi equation then the residual angle dependence, V_1, would be zero.

The two terms are most easily calculated as integrals over the original angle variable Φ_0 using the transformation $\Phi_1 = \Phi_0 + G^1_I(\Phi_0, I_1, s)$ and its Jacobian. We group all terms in the Hamiltonian transformation equation that might have angular dependence into

$$F(\Phi_0, I_1, s) = G^1_s(\Phi_0, I_1, s) + \frac{G^1_\Phi}{\beta(s)} + V(\Phi_0, I_1, s) \quad . \quad (4)$$

Then A is the average of F over the new angles; this can be written in the original angles, by using the Jacobian, as

$$A(I_1, s) = \int_0^{2\pi} \frac{d\Phi_0}{2\pi}\, \mathcal{J}(\Phi_0, I_1, s)\, F(\Phi_0, I_1, s) \quad . \quad (5)$$

where $\mathcal{J}(\Phi_0, I_1, s) = 1 + G^1_{I\Phi}(\Phi_0, I_1, s)$. In a similar fashion, the Fourier coefficients of V_1 with respect to the *new* angles Φ_1 are calculated as

$$V_1(m, I_1, s) = \int_0^{2\pi} \frac{d\Phi_0}{2\pi}\, \mathcal{J}(\Phi_0, I_1, s) \times e^{-im(\Phi_0 + G^1_I)}\, F(\Phi_0, I_1, s) \quad . \quad (6)$$

The Φ_1 dependence can be found using the Fourier series $V_1(\Phi_1, I_1, s) = \sum_{m \in M_1} V_1(m, I_1, s)\, e^{im\Phi_1}$, where M_1 is a mode set bigger than the original M_0. For the second perturbative transformation it is the Fourier coefficients themselves that are interesting.

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† Current mailing address is SLAC, Bin 26, P. O. Box 4349, Stanford, CA 94309.

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TABLE 1. The results for refined solutions and their comparison with non-perturbative solutions.

action I_2	with refinement				cpu	non-perturbative only			cpu
(10^{-5} m)	M_0	N_{RK}	M_1	δ	(secs)	M_0	N_{RK}	δ	(secs)
2.2	31	6→28	127	$1.79{\cdot}10^{-2}$	244	63	14→28	$4.10{\cdot}10^{-2}$	468
2.0	31	6→20	127	$2.09{\cdot}10^{-3}$	177	63	10→20	$3.67{\cdot}10^{-3}$	314
1.0	31	6→20	127	$5.52{\cdot}10^{-4}$	156	63	10→20	$5.41{\cdot}10^{-4}$	436
0.1	7	2→12	31	$4.98{\cdot}10^{-5}$	5.3	31	6→12	$5.05{\cdot}10^{-5}$	40.5
0.01	3	1→4	31	$9.45{\cdot}10^{-6}$	2.0	7	2	$1.78{\cdot}10^{-4}$	0.4

II. The Second, Perturbative Canonical Transformation

Using the new Hamiltonian (3) with the residual angle dependence, we can write the Hamilton-Jacobi equation that gives the transformation to a new set of variables (Φ_2, I_2). If the G^1 canonical transformation was an approximate solution to the original Hamilton-Jacobi equation then the residual perturbation will be much smaller than the original perturbation. It then makes sense to solve the new Hamilton-Jacobi equation with perturbative techniques.

The Hamilton-Jacobi equation for the generator $G^2(\Phi_1, I_2, s)$ is

$$H_2(I_2,s) \simeq G^2_s + \frac{I_2 + G^2_\Phi(\Phi_1, I_2, s)}{\beta(s)} + A(I_2,s) + \partial_I A(I_2,s)\, G^2_\Phi + V_1(\Phi_1, I_2, s) \quad . \quad (7)$$

In Equation (7) we have expanded to first order in the presumed small parameters V_1 and G^2. The above equation can be solved by using a Fourier series for G^2 of the same form as Equation (2) with Fourier coefficients $g^2(m, I_2, s)$ and summing over $m \in M_1$.

The Hamiltonian (3) defines a betatron phase advance that is slightly different from the linear phase advance. We define the new phase as

$$P(s) = \int_0^s \frac{d\sigma}{\beta(\sigma)} + \int_0^s \partial_I A(I_2, \sigma)\, d\sigma \quad . \quad (8)$$

Then the Fourier coefficients of $G^2(\Phi_1, I_2, s)$ can be written as

$$g^2(m,I_2,s) = -e^{-i\,m\,P(s)}\{\frac{1}{e^{i\,m\,P(C)} - 1} \times \int_0^C e^{i\,m\,P(\sigma)} V_1(m, I_2, \sigma) d\sigma + \int_0^s e^{i\,m\,P(\sigma)} V_1(m, I_2, \sigma) d\sigma\} \quad , \quad (9)$$

recalling that $m \in M_1$ and that I_2 is a constant parameter.

The original action I_0 can be specified as a function of the original angle Φ_0 by chaining through the two canonical transformations. After expanding all functions of $I_1 = I_2 + G^2_\Phi(\Phi_1, I_2, s)$ around the 'best' constant action I_2, we find the compact form

$$I_0(\Phi_0, I_2, s) = I_2 + G^1_\Phi(\Phi_0, I_2, s) + \det\left(\frac{\partial I_0}{\partial I_1}\right) G^2_\Phi(\Phi_0 + G^1_I, I_2, s) \quad . \quad (10)$$

This is the refined invariant torus. It gives the distortions to the linear Courant-Snyder actions under the effects of the nonlinearity.

III. Numerical Results

The accuracy of the invariant torus can be estimated by finding its deviation from numerically computed trajectories. The trajectories are computed by a fourth order symplectic integrator[2] . The deviations of the actions as each trajectory crosses the $s = 0$ point (the point at which we are studying the torus) from the corresponding actions of the torus are found. The normalized deviation is calculated as

$$\delta = \max \frac{\sum_{k=1}^{1000} |I_0(\Phi_0^T(kC)) - I_0^T(kC)|}{\sum_{k=1}^{1000} |I_0(\Phi_0^T(kC)) - I_2|} \quad , \quad (11)$$

where superscript T indicates points on the trajectory, and where the summation is over the turn number. The maximum is over 16 trajectories with different initial conditions starting on the torus. The δ parameter measures the worst agreement between a trajectory and the torus. Notice it is normalized by the distortion of the torus and not by the action value itself.

In Table 1 we compare the refined solution to a good non-perturbative solution. We give the mode sets and the number of integration steps, N_{RK}, for the fourth order Runge-Kutta integration used. The final number of integration steps plus one is the number of knots used for the cubic spline interpolation of the s dependence of the Fourier coefficients g^1 and g^2 in a sextupole. The s dependence outside the nonlinear elements is trivial. The cubic spline is found using the 'not-a-knot' condition, and it is used to evaluate the integrals in Equation (9). The 'CPU' is the computation time to calculate the solution on the SLAC IBM 3090. The time under 'with refinement' gives the time for finding the poor

non-perturbative canonical transformation, for estimating $\partial_I g^1$, and for completing the perturbative calculation. The time under 'non-perturbative only' is that for calculating just the good non-perturbative solution.

The numerical results are for the ideal single cell of the Berkeley Advanced Light Source (ALS)[3] . The largest amplitude in Table 1 corresponds to an approximate x offset of 22 mm, and the smallest to 1.5 mm. The x tune is $\nu = 1.18973$. There are four 0.20 m long sextupoles symmetrically placed in the cell with strengths of -88.09, 115.615, 115.615, and -88.09 m^{-3}.

In Figure 1, we display three representative invariant curves. The top curve corresponds to $I_2 = 2 \cdot 10^{-5}$ m and the bottom to $I_2 = 10^{-6}$ m. On this plot, the difference between the original and the refinement to an invariant torus is not discernible. If there were no nonlinearity, the curves would be flat lines at their respective values of I_2.

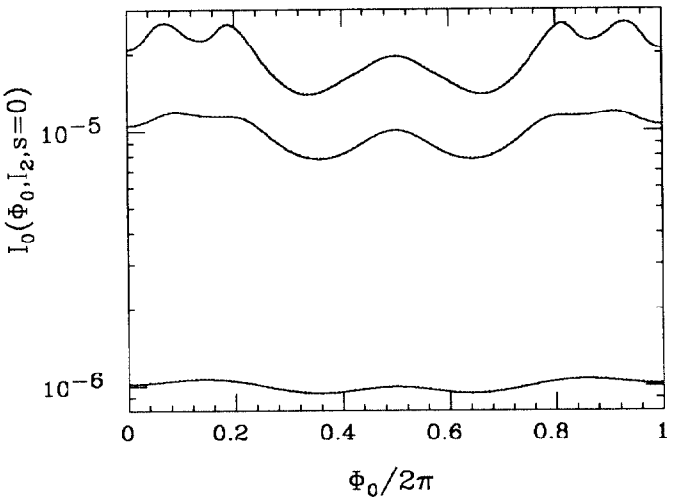

Figure 1. Invariant curves shown at $s = 0$ for the constant actions $I_2 = 2 \cdot 10^{-5}$ m, 10^{-5} m, and 10^{-6} m. The refinement is not noticeable on this plot.

From the first non-perturbative transformation, we calculate the function A which defines the nonlinear phase advance, $P(s) - \Psi(s)$, where $\Psi(s) = \int_0^s d\sigma/\beta(\sigma)$ is the linear phase advance. Notice that it varies only in the nonlinear elements. In Figure 2, we show the nonlinear part of the phase advance in each of the four sextupoles. On the horizontal axis in the plot, 1 to 2 corresponds to the first sextupole, 2 to 3 the second, and so on. The offset from zero at the end of the last sextupole is 2π times the nonlinear tune shift.

In Figure 2, notice that the shape of the function $P(s) - \Psi(s)$, is very similar for the three cases shown even though the magnitudes are different. This is typical for this accelerator example. The three cases have constant actions $I_2 = 10^{-6}$ m to $2 \cdot 10^{-5}$ m and correspond to the invariant curves shown in Figure 1.

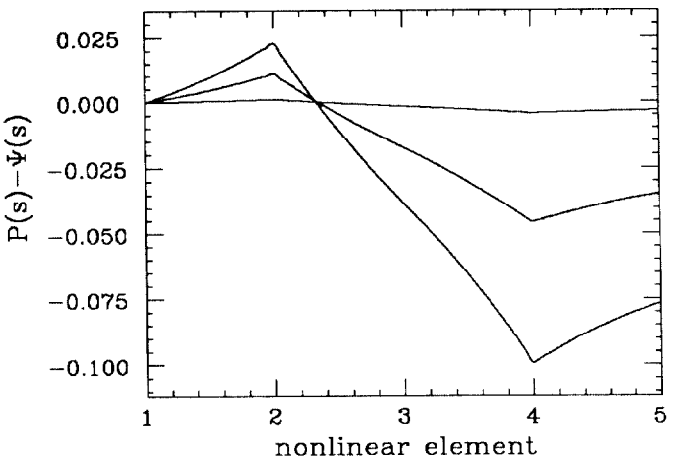

Figure 2. Nonlinear phase advance in the sextupoles for $I_2 = 10^{-6}$ m, 10^{-5} m, and $2 \cdot 10^{-5}$ m. The horizontal axis is 1 to 2 for the first sextupole, 2 to 3 for the second and so on. The small offset at the end of the last sextupole is 2π times the nonlinear tune shift. The linear phase advance is $\Psi(s) = \int_0^s d\sigma/\beta(s)$.

IV. Conclusions

We found that the second canonical transformation used to refine a poor non-perturbative solution gave more accurate tori than the poor solution. However the refinement of very good non-perturbative solutions did not increase the accuracy of the solutions. From Table 1 we see that in most cases the refinement gives tori with similar accuracy in about half the computation time. From the implementation of the non-perturbative transformation, we were able to calculate the nonlinear phase advance in the sextupole magnets. The form that these functions take seems to depend on only a relatively small number of parameters as the constant action I_2 is changed.

V. References

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