CONSTRUCTION OF HIGH ORDER MAPS FOR LARGE PROTON ACCELERATORS*

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Abstract

We propose a simple perturbation method using a truncated CBH Theorem to calculate the nonlinear generator for one turn maps, and argue that such maps should give accurate long turn dynamic apertures for proton colliders such as the SSC and LHC.

I. INTRODUCTION

A promising application for maps is the determination of long term dynamic apertures. Since the exact machine is not known, a statistical ensemble must be studied to establish confidence in design parameters and specifications. This requires studying a large set of statistically generated machines. To use maps to study dynamic apertures one must be able to rapidly track them and construct them. The rapid tracking of maps may be done through a kick factorization [1] or a Fourier developed generating function. [2] Here we address calculation of maps.

The present method of calculating high order maps requires tracking a Taylor series element-by-element around the ring to determine the one turn Taylor series map. [3] At interesting orders the CPU time to construct the map in this way becomes prohibitive for a large machine. For example a ninth order SSC map requires more than two hours of Cray CPU time.

We propose here a method by which these maps can be calculated rapidly. The method also provides insight into the physical sources of terms in the map.

II. DESCRIPTION OF THE METHOD

A. *The Lattice Representation*

Most element-by-element tracking programs begin by approximating the lattice by a series of linear transformation and nonlinear kicks. Canonical integrators prescribe the magnitude and sequence of kicks for thick elements. [4] The linear transformations may be analytically represented by matrices and the kicks may be represented by exponential Lie operators. [5] There are also ways to represent a thick element by two linear transformations with an exponential Lie operator acting at the center. [6] Errors in placement and strength as well as fringe fields can be represented this way. Consequently, a product of linear transformations and nonlinear exponential Lie operators is a very general starting point for a lattice representation.

Using a similarity transformation,

$$Re^{:f:}R^{-1} = e^{:Rf:},$$

where R is a linear operator, one can rearrange operations to obtain a product of linear transformations times a product of Lie operators.

$$\prod_{1\leq n\leq N} e^{:f_n:}R_n = \prod_{1\leq n\leq N} R_n \prod_{1\leq n\leq N} e^{:\bar{R}_n f_n:}$$

where $\bar{R}_n = \prod_{1\leq m\leq n} R_m$.

B. *CBH Theorem*

The nonlinear problem becomes a problem in concatenating the Lie operators. Two exponential Lie operators may be combined into one by using the Cambell-Baker-Hausdorf (CBH) Theorem.

$$e^{:a:}e^{:b:} = e^{:c:}$$

with

$$c = a + b + \frac{1}{2}[a,b] + \frac{1}{12}\left[a-b,[a,b]\right] + \frac{1}{24}\left[b,[a,[a,b]]\right] + \ldots$$

To combine several exponential Lie operators one can start at one end and proceed stepwise. At the nth step,

$$e^{:f_{n+1}:}e^{:F_n:} = e^{:F_{n+1}:}$$

where F_n is the generator corresponding to the product of the first n Lie operators.

The CBH theorem can be understood as a perturbation series. The initial sum of generators, $c \approx a+b$, gives the correct result when there is no interaction between the nonlinearities. The first Poisson bracket (PB), $\frac{1}{2}[a,b]$, gives the first order (in generator strength) modification as the action of one operator is altered by the presence of the other, corresponding to the modification of one kick that results from a previous kick in a ring. The double PB gives second order effects and so on. Thus one may view the CBH theorem for the product generator as a perturbation series in the nonlinearity strength. If each nonlinearity is small, which is indeed the case with proton colliders, the higher order terms in the perturbation series are quite small.

It is important to note that this perturbation series is not the same as truncating at some polynomial order. The individual nonlinearities can contain high order monomials.

C. *Separation of Orders*

There may be occasion or interest in keeping track of contributions that come from different orders. In the method described above terms of different order become mixed. This is not necessary, and with a bit more work, terms of different order can be kept separate. For this purpose we will replace f_n by εf_n and expand F_n.

$$F_n = \varepsilon F_n^1 + \varepsilon^2 F_n^2 + \varepsilon^3 F_n^3 + \ldots$$

Expanding the CBH theorem in powers of ε and identifying terms of equal power one finds:

$$F_{n+1}^1 = f_{n+1} + F_n^1$$

$$F_{n+1}^2 = F_n^2 + \frac{1}{2}[f_{n+1}, F_n^1]$$

$$F_{n+1}^3 = F_n^3 + \frac{1}{2}[f_{n+1}, F_n^2] + \frac{1}{12}\left[f_{n+1} - F_n^1, [f_{n+1}, F_n^1]\right]$$

$$F_{n+1}^4 = F_n^4 + \frac{1}{2}[f_{n+1}, F_n^3] + \frac{1}{12}\left[f_{n+1}, [f_{n+1} - F_n^1, F_n^2]\right] + \frac{1}{24}\left[F_n^1, [f_{n+1}, [f_{n+1}, F_n^1]]\right].$$

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Using these relationships it is possible to write the F_n's directly in terms of the f_n's. For example,

$$F_m^1 = \sum_{n \le m} f_n$$
$$F_m^2 = \sum_{p \le n \le m} \frac{1}{2}[f_n, f_p].$$

III. Estimating and Calculating Poisson Brackets

A. Estimating Poisson Brackets (PBs)

From the basic definition of the exponential operator,

$$z' = e^{:a:} z = z + [a, z] + \frac{1}{2}[a, [a, z]] + \dots$$

it follows that for a weak nonlinearity $\delta z \approx [a, z]$. For an action variable $J \equiv 0.5(x^2 + p_x^2)$, $\delta J \approx [a, J]$.

For a a polynomial of order p, which we denote by a_p, let us take as an estimate of the strength of a_p the expression

$$a_p^{\#} \equiv \max_{0 \le \theta \le 2\pi} \frac{|\delta J|}{2J}$$

We have included the factor 2 because J is quadratic in phase space variables x and p_x, and defined this way the estimate is related to the definition of smear [7]. Writing x and p_x in terms of J and θ we find

$$a_p = \sum_k \alpha_{pk} J^{\frac{p}{2}} e^{ik\theta}$$

where α_{pk} is complex, $\alpha_{p,-k}$ is the complex conjugate of α_{pk}, and only even or odd values of k occur according to whether p is even or odd. Thus

$$\frac{\delta J_a}{J_a} \approx \sum_k ik\alpha_{pk} J^{\frac{p}{2}-1} e^{ik\theta}.$$

As a rough estimate of the sum, we replace each quantity by its average value and multiply by the number of turns. Then

$$\frac{|\delta J_a|}{J} \approx \frac{p^2}{2}\langle|\alpha_{pk}|\rangle J^{\frac{p}{2}-1}$$

Likewise for b_q

$$b_q = \sum_j \beta_{qj} J^{\frac{q}{2}} e^{ij\theta}$$

and

$$\frac{|\delta J_b|}{J} \approx \frac{q^2}{2}\langle|\beta_{qj}|\rangle J^{\frac{q}{2}-1}$$

To estimate $\delta J_{[a_p,b_q]}$ we first calculate $[a_p, b_q]$.

$$[a_p, b_q] = J^{\frac{p+q}{2}-1} \sum_{j,k} \alpha_{p,k}\beta_{q,j}\, i\frac{pj - qk}{2} e^{i(j+k)\theta}$$

From $\langle|(pj - qk)(k + j)|\rangle \approx \max(p, q)pq/3$ and the same estimation procedure as above, we get

$$\frac{|\delta J_{[a,b]}|}{J} \approx J^{\frac{p+q}{2}-2} \frac{\max(p,q)(pq)^2}{6} \langle|\alpha_{p,k}|\rangle\langle|\beta_{q,j}|\rangle$$

from which we deduce

$$\frac{|\delta J_{[a,b]}|}{J} \approx \frac{2}{3}\max(p,q)\frac{|\delta J_a|}{J}\frac{|\delta J_b|}{J}.$$

or

$$\left(\frac{1}{2}[a_p, b_q]\right)^{\#} \approx \frac{2}{3}\max(p,q) a_p^{\#} b_q^{\#}$$

Another way to look at this is to make use of the Jacobi identity to write

$$\delta J_{[a,b]} \approx [[a,b], J] = [a, [b, J]] - [b, [a, J]] = [a, \delta J_a] - [b, \delta J_b]$$

where it is now clear that the change of J due to $[a, b]$ is the change of a change, and hence of higher order. The factor $2\max(p, q)/3$ comes from the fact that the derivative of the change comes in, not just its magnitude. We have used the fact that for polynomials of known order we can estimate the derivative in terms of the function.

For the SSC, at values of J near the long term dynamic aperture $\frac{|\delta J_f|}{2J} \approx .001$ for an f generating the nonlinearities of a single dipole. As one proceeds around the ring the cumulative effects of these nonlinearities grows as $\sqrt{N}$, where N is the number of dipoles. For the generator representing the nonlinearities of the full ring $\frac{|\delta J|}{2J} \approx .06$ at J near the long term dynamic aperture. We note that the long term dynamic aperture is very near what is referred to as the linear aperture, the latter being defined as that aperture with smear equal 10%.

The connection between the nonlinear generator we are calculating and the generator giving the smear is to be found in Normal Form theory. [8] Normal Form theory attempts to combine the linear transformation and the nonlinear factors into one grand generator which would be the pseudo-Hamiltonian of the ring. This is a risky procedure which works in some situations and not in others. Resonance denominators are involved. Such problems do not arise in the formation of the nonlinear generator we are constructing.

B. Calculation of Poisson Brackets

A PB in two degrees of freedom can be calculated by taking four derivatives of each function and performing four multiplications. In the general situation it is preferrable to use the PB for the basic monomials:

$$\left[\alpha_{abcde} x^a p_x^b y^c p_y^d \delta^e, \beta_{pqrst} x^p p_x^q y^r p_y^s \delta^t\right] = \alpha_{abcde}\beta_{pqrst}\delta^{e+t}$$

$$\{(aq - bp)x^{a+p-1}p_x^{b+q-1}y^{c+r}p_y^{d+s} + (cs - dr)x^{a+p}p_x^{b+q}y^{c+r-1}p_y^{d+s-1}\}$$

The number of operations involved is somewhat greater than two multiplications if the monomials are scanned to optimize computation time. When one of the functions is a kick the calculation can be reduced to two multiplications. This follows from the relationship

$$\left[\sum \alpha_{nm} x^n y^m, F\right] = \sum n\alpha_{nm} x^{n-1} y^m [x, F] + \sum m\alpha_{nm} x^n y^{m-1} [y, F].$$

C. Comparison with Power Series Tracking

When carefully optimized the number of polynomial multiplications per dipole in power series tracking is about twice the order retained in the error multipoles of the dipole. For 5th order multipoles (through dodecapole) this is 12 multiplications per dipole. For the procedure we are suggesting the number of PBs will be one or two per dipole corresponding to approximately two or five multiplications, independent of maximum multipole order. For ninth order multipoles the improvement in computational time could be almost a factor of ten. Some computations will be needed to establish the exact improvement because there are other effects: the order of the generators is larger by one than the order of the map, though they begin at third order for the first PB and at fourth order for the second PB and many pairs are absent.

IV. Choice of Truncation of the CBH Theorem

A. *Truncation Order for Ensembles of Maps*

In concatenating the many weak nonlinear generators into one generator it is crucial to maintain at least one of the PBs in the CBH theorem. This conclusion rests on the fact that one PB is needed to include the effects of sextupoles acting on sextupoles. Such terms produce an octupole like term that gives an important contribution to the phase shift with amplitude.

However there is an indication that statistically this may be sufficient. Statistical analytic calculations of tune shift and smear, including one PB for the tune shift and none for the smear, have been compared with tracking over 400 seeds. The tracking results were Fourier analyzed so that one could compare smear details. The results agreed [9] to within the accuracy of the tracking analysis.

One might further argue that if you have the correct tune shift with amplitude, which determines the position of resonance lines in the particle phase space, and the correct smear amplitudes, which determine the driving strengths of resonances, the long term dynamic apertures would also agree. We believe this will be especially true when one compares ensembles of machines because the coefficient of any particular monomial in the map will vary more widely with the change in the first order terms than with modification by higher order effects. If this is indeed the case, the limits on our ability to know and measure the high order multipoles in a single magnet will limit our ability to predict the long term dynamic aperture.

B. *Statistical Generation of Maps*

If it can be established that the maps constructed with one PB produce the same long term dynamic aperture, or even that statistically they give the same long term dynamic aperture, then it is possible to generate these maps analytically. [9] This can be done extremely fast. In this case it should be possible to identify the important coefficients in the map and their source. This would provide important insight into the determination of the long term dynamic aperture.

C. *Truncation Order for Individual Maps*

Though we believe it is enough to include one PB of the CBH theorem when studying the properties of an ensemble of machines, we recognize it would build confidence in the method to show that the map for a single particular machine generated in the way we propose gives exactly the same dynamic aperture as the element-by-element tracking of that machine. For this purpose we conjecture that one more PB may be enough. This belief rests on our estimation of the accuracy of the approximation at this level, that the tune shift with amplitude and the resonant strengths are determined accurately, and also on the fact that the next order PB, if it were to be included, is even an order smaller than would be expected at first sight. In other words including one more PB includes all effects that occur in the next two orders of approximation. We now present the argument for this result.

Denote by f_n the nth nonlinear generator, and by F_n the generator that combines the first n generators. For the SSC the result of summing up terms $[f_{n+1}, F_n]$ is about $[F_N, F_N]$, and so is approximately of magnitude $(.06)^2$ at the long term dynamic aperture. The result of summing a term like $[F_n, [f_{n+1}, F_n]]$ will be down by another factor of .06 because it involves an additional PB. However the term $[f_n, [f_n, F_n]]$ will be smaller yet by a factor of 1/50 because F_n grows like $\sqrt{n}$ compared to f_n, and half way around the ring $\sqrt{n} \approx 50$. Thus at any given order of PB, terms which have two f's will be smaller than terms with one f by more than one order.

Since the only third order PB occurring in the CBH theorem is

$$[f_{n+1}, [F_n, [f_{n+1}, F_n]]]$$

and since it has two f's, effects from this order will be more than two orders smaller than the second order PB terms.

V. Conclusion

We have described a method to calculate one turn maps that offers insight into the structure of the maps and is substantially faster than standard power series methods. If the program suggested here is carried out and the conjectures proven correct, the nonlinear generators for one turn map maps could be calculated directly using analytic expressions containing random numbers. Substantial insight would be gained into the physics behind the determination of long term dynamic apertures.

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