Particle Amplitude Growth Due to Single or Repetitive Resonance Crossing

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We study the problem of the growth of the amplitude of a particle executing betatron oscillations in a synchrotron or storage ring, due to the crossing of a resonance line. The resonance can be crossed either only once, or repeatedly. We treat only one degree of freedom in this paper, and assume the particles in the beam are independent. We consider some simple models of amplitude-dependent tunes, to discern the effects of detuning away from the resonance as the particle amplitude grows. The basic equation of motion we shall treat is the harmonic oscillator equation

$$\ddot{x} + \omega_x^2\, x \;=\; f(t) \tag{1}$$

where the frequency $\omega_x = \omega_x(x,t)$ may depend on x and t, and also other parameters, and f is a driving term, which may also depend on t and other quantities. A more complete description of our analysis and results, including numerical results, is given in Ref. [1]. We only present the main points here. We shall assume that the driving term f has a sinusoidal time dependence, i.e. $f \propto F\cos(\omega_f t)$, or sometimes $f \propto F e^{i\omega_f t}$. First, let us suppose that $\omega_x = \omega_{x0}$, a constant, and

$$\ddot{x} + \omega_{x0}^2\, x \;=\; \frac{F}{2\tau_{rev}} e^{i\omega_f t}\,, \tag{2}$$

where τ_{rev} is the revolution time around the ring. The solution for the driven part of the motion is easily found to be

$$x \;=\; -i\,\frac{F}{2\tau_{rev}}\,\frac{e^{i\omega_f t}}{\omega_{x0}^2 - \omega_f^2} \tag{3}$$

and, on resonance, i.e. $\omega_{x0} = \omega_f$, the solution is

$$x \;=\; -i\,\frac{F}{2\tau_{rev}}\,\frac{t\,e^{i\omega_f t}}{2\omega_f}\,. \tag{4}$$

This grows without bound as $t \to \infty$. Suppose, however, there is an amplitude dependent tuneshift, given by

$$\omega_x \;=\; \omega_{x0} + \mu I_x\,, \tag{5}$$

where μ is a constant, and $I_x = [(\omega_{x0}x)^2 + \dot{x}^2]/(2\omega_{x0})$ is the action. The above model is an octupole tuneshift, since it is quadratic in the particle amplitude, i.e. linear in I_x. Suppose also that the motion is on resonance in the absence of the tuneshift, i.e. $\omega_f = \omega_{x0}$. Then the motion is resonant for small values of I_x, but as I_x increases, the motion is detuned, i.e. it goes off resonance. Putting $x = u\,e^{i\omega_f t}$, we find that, from the above definition, $I_x = \omega_{x0}u^2/2$. We also find that the motion does not grow to infinity, but reaches a peak value, given by [1]

$$u_{max} \;=\; \left(\frac{F}{2\tau_{rev}\mu\omega_{x0}^2}\right)^{1/3}, \tag{6}$$

$$I_{max} \;\propto\; u_{max}^2 \;\propto\; \left(\frac{F}{\mu}\right)^{2/3}. \tag{7}$$

We have verified the above result by tracking [1].

We now treat the next case in our investigation. We now suppose that ω_x is a constant, without any amplitude dependence. Suppose also that the frequency of the driving term increases linearly with time, i.e. $\omega_f(t) = \omega_1 + \dot{\omega}t$, where ω_1 and $\dot{\omega}$ are constants. Without loss of generality, we may suppose that $\omega_1 = \omega_x$ and that $\dot{\omega}$ is positive. In that case, the frequency of the driving term starts off below ω_x, then becomes resonant with ω_x, and then again becomes nonresonant. We again expect that the particle amplitude will grow, to a large but finite asymptotic value, but the expression for the peak amplitude will be different from the previous case. The problem can in fact be solved analytically. We write

$$\phi(t) \;=\; \int_0^t \omega_f(t')\,dt' \;=\; \omega_1 t + \frac{\dot{\omega}t^2}{2} \tag{8}$$

and, using a Green function approach, we can show that

$$\begin{aligned} x \;&=\; \frac{1}{\omega_x}\int_{-\infty}^{t} \sin[\omega_x(t-t')]F\sin[\phi(t')]\,dt' \\ &=\; -\frac{F}{2\tau_{rev}\omega_x}\left(\frac{2\pi}{\dot{\omega}}\right)^{1/2}\cos(\omega_x t + \frac{\pi}{4})\,, \end{aligned} \tag{9}$$

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after some algebra. The peak amplitude is therefore

$$|\omega_x^2 x^2 + \dot{x}^2|^{1/2} = \frac{F}{2\tau_{rev}}\left(\frac{2\pi}{\dot{\omega}}\right)^{1/2} \propto \frac{F}{\dot{\omega}^{1/2}}. \quad (10)$$

Lysenko [2] also found, independently, that the parameter which characterizes the amplitude growth is $F/\dot{\omega}^{1/2}$. We can offer a heuristic derivation of the above result. In the vicinity of the resonance, the amplitude of x grows linearly with time, $x \propto t$, so the amplitude growth is approximately proportional to the time spent near the resonance, say T. Now the particle will be close to the resonance when $\omega_f \simeq \omega_x$, so that the phase of the particle oscillation, $\omega_x t$, and of the driving term, $\int \omega_f \, dt'$, are in phase. However, ω_f increases with time, so after a while $\omega_x t$ and $\int \omega_f \, dt'$ will get out of phase. This will happen, roughly speaking, when the phase gap between them reaches 2π. Hence the time T is approximately given by

$$\int_0^T \omega_f \, dt' \simeq \omega_x T + 2\pi. \quad (11)$$

Putting $\omega_f = \omega_x + \dot{\omega}t'$, we obtain

$$\frac{\dot{\omega}T^2}{2} \simeq 2\pi \quad (12)$$

or $T \propto \dot{\omega}^{-1/2}$. As argued above, the particle amplitude growth is proportional to T, hence $\dot{\omega}^{-1/2}$. The factor of F is obvious because the equation of motion is linear, hence $x \propto F/\dot{\omega}^{1/2}$.

We now turn to the third case to be studied in this paper, viz. multiple, or repetitive, crossing of a resonance. This type of situation is more pertinent to the "storage mode," where the revolution frequency, betatron frequency, and driving frequency are fixed for long periods of time. Because of synchrotron oscillations and chromaticity, the betatron frequency of a particle may oscillate sinusoidally and cross a resonance line repeatedly. Let us therefore suppose that

$$\omega_x = \omega_{x0} + \lambda \cos(\omega_m t) \quad (13)$$

where λ is the amplitude of the tune modulation and ω_m is the modulation frequency. We refrain from writing ω_s instead of ω_m because the above modulation may be caused by other sources, e.g. power supply ripple, not only synchrotron oscillations. We do, however, assume that $\omega_m \ll \omega_{x0}$ and $\omega_m \ll \omega_{rev}$, the (angular) revolution frqeuency. The solution for x, using a Green function approach, can be written as

$$x = \frac{F}{\omega_{x0}} \int_{-\infty}^{t} \sin\left(\int_{t'}^{t} \omega_x \, dt''\right) \sin(\omega_f t') \, dt'. \quad (14)$$

We use the Bessel function identity

$$e^{ir\sin v} = \sum_{n=-\infty}^{\infty} e^{inv} J_n(r). \quad (15)$$

We also average over the betatron oscillations, and treat only the centroid $\overline{x}$. The solution can then be written as [1]

$$\overline{x} = \int_{-\infty}^{t} \frac{F(t')}{\omega_{x0}} e^{-\overline{\mu}(t-t')} \sin\Big[\omega_{x0}(t-t') + \frac{\lambda}{\omega_m}[\sin(\omega_m t) - \sin(\omega_m t')]\Big] \frac{dt'}{\tau_{rev}} \quad (16)$$

A convergence factor $e^{-\overline{\mu}(t-t')}$ has been introduced into the above expression, to model the decoherence caused by an amplitude dependent tune. After some algebra, the solution is

$$\overline{x} = \frac{F}{2\omega_{x0}\tau_{rev}} \mathrm{Re}\Big\{ e^{i(\lambda/\omega_m)\sin(\omega_m t)} \times \sum_n e^{-in\omega_m t} J_n\left(\frac{\lambda}{\omega_m}\right) \times \Big[\frac{e^{i\omega_f t}}{\overline{\mu} - i(\omega_{x0} + n\omega_m - \omega_f)} - \frac{e^{-i\omega_f t}}{\overline{\mu} - i(\omega_{x0} + n\omega_m + \omega_f)}\Big]\Big\}. \quad (17)$$

In other words, there are resonances at the betatron frequency ω_{x0} plus "sidebands" $\omega_{x0} + n\omega_m$, $n = 0, \pm 1, \pm 2, \ldots$

There are now two cases to consider. If the sidebands are very close, i.e. $\omega_m \ll |\omega_f - \omega_{x0}|$, then we find [1]

$$\overline{x} = \frac{F}{2\omega_{x0}\tau_{rev}} \mathrm{Re}\Big\{ \frac{e^{i\omega_f t}}{\overline{\mu} - i(\omega_{x0} - \omega_f)} - \frac{e^{-i\omega_f t}}{\overline{\mu} - i(\omega_{x0} + \omega_f)} \Big\}. \quad (18)$$

This is exactly as if there had been no tune modulation at all, as can be verified by putting $\lambda = 0$ in the integral for $\overline{x}$. Hence if the driving frequency lies outside the "cluster" of the betatron frequency and its sidebands, then the cluster can be treated as one single frequency.

If the sidebands are widely separated, so that ω_m is comparable to $|\omega_{x0} - \omega_f|$, then we approximate by assuming that only one sideband (only one value of n) is important, in which case

$$\overline{x} \propto \frac{F}{2\omega_{x0}\tau_{rev}} \frac{J_n(\lambda/\omega_m)}{\overline{\mu} - i(\omega_{x0} + n\omega_m - \omega_f)}. \quad (19)$$

If the tune modulation is due to synchrotron oscillations, then we must average over the amplitude and phase of the synchrotron oscillations. The details are given in Ref. [1] and are not relevant here.

In conclusion, we have presented three case studies of situations where the amplitude of a betatron oscillation can grow, due to crossing a resonance. We have presented expressions for the amplitude growth caused by the resonance, in the various cases. Further investigations, e.g. involving two degrees of freedom, or other types of driving terms and other types of resonances, will be presented elsewhere.

References

[1] S.R. Mane and W.T. Weng, Nucl. Instr. and Meth., in press.

[2] W.P. Lysenko, Part. Accel. **5**, 1 (1973).