Grid Scans: A Transfer Map Diagnostic*

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ABSTRACT

A beam line transfer map diagnostic is described which uses induced betatron oscillations to search for focusing errors and geometric aberrations. A grid is produced graphically in *normalized phase space* coordinates with the beta match quantified from this grid. Application to the SLC electron damping Ring-To-Linac (RTL) transport line is presented.

I. DATA ACQUISITION

Two horizontal (or vertical) dipole corrector magnets are scanned in *nested loop* fashion to induce cosine-like and sine-like betatron oscillations in the RTL. Beam Position Monitors (BPM) of the RTL are sampled for each corrector setting and this data is saved to disk. BPM readings are averaged over 5 beam pulses to reduce noise levels. The corrector scan range is selected to produce oscillations which explore ~8 times the monochromatic RMS beam size.

II. ANALYSIS

The linear transformation used to obtain normalized phase space coordinates from a pair of BPMs is

$$\vec{u}_1 \equiv \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{\varepsilon\beta_1} & 0 \\ \alpha_1/\sqrt{\varepsilon\beta_1} & \sqrt{\beta_1/\varepsilon} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ R_{11}^{(1:2)} & R_{12}^{(1:2)} \end{bmatrix}^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} , \quad (1)$$

where $\mathbf{R}^{(1:2)}$ is the 2×2 transfer matrix from BPM-1 to BPM-2, α_1 and β_1 are design Twiss parameters at BPM-1, ε is the nominal RTL beam emittance (6800 μm-μrad), u_1,v_1 are the desired normalized phase space coordinates at BPM-1, and x_1, x_2 are the position readings from a BPM pair. These BPMs are separated by one to optimize phase advance (Fig 1). The mean of all trajectories is subtracted to produce difference trajectories.

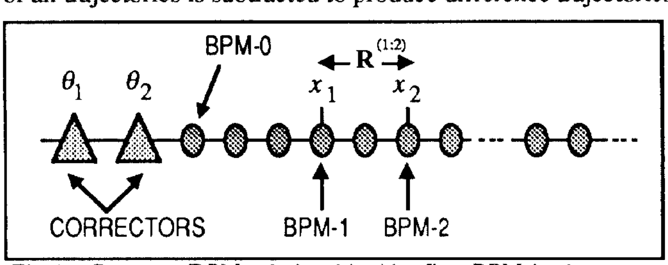

Fig 1. Corrector/BPM relationship (the first BPM in the system, BPM-0, and an arbitrary BPM pair is shown as an example - x_1, x_2)

Including the emittance in the normalization conveniently scales the *u*,*v* coordinates to units of nominal RMS beam size. This transformation is applied to each point per BPM pair and v_1 is plotted versus u_1 to form a grid of points. The grid construction is repeated for each BPM pair in the beam line.

The transfer matrices are taken from the SLC Data Base after an on-line COMFORT [1] run. The Twiss parameters used are the propagation of some chosen initial Twiss parameters, α_0 and β_0 through the modelled transfer matrix elements.

$$\begin{bmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{bmatrix} = \begin{bmatrix} R_{11}^2 & -2R_{11}R_{12} & R_{12}^2 \\ -R_{11}R_{21} & 1+2R_{12}R_{21} & -R_{12}R_{22} \\ R_{21}^2 & -2R_{21}R_{22} & R_{22}^2 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{bmatrix} \quad (2)$$

Here the **R** matrix transports from some chosen initial fixed point, BPM-0 in Fig 1, to the first BPM of the current pair of interest (BPM-1). The initial Twiss parameters are chosen so that the first BPM pair produces a square grid (see section III). The choice of correctors is then less restricted. It is only necessary that they sufficiently span the space ($\Delta\psi \neq n\pi$). A linear grid is fitted to the transformed data points using the known corrector kick angles θ_1 and θ_2 as a parametrization.

$$u_1(t) = a\theta_1(t) + b\theta_2(t)$$
$$v_1(t) = c\theta_1(t) + d\theta_2(t) \quad (3)$$

The fit coefficients (a, b, c, and d) are used only to evaluate a fitted u_1 and v_1 at each of the grid points, therefore the absolute scale of the kick angles is irrelevant. The transformed data points and the fitted points are then superimposed on the same plot. The fitted points are connected by lines to form the grid, and the transformed raw BPM data points are represented as dots (Fig 2).

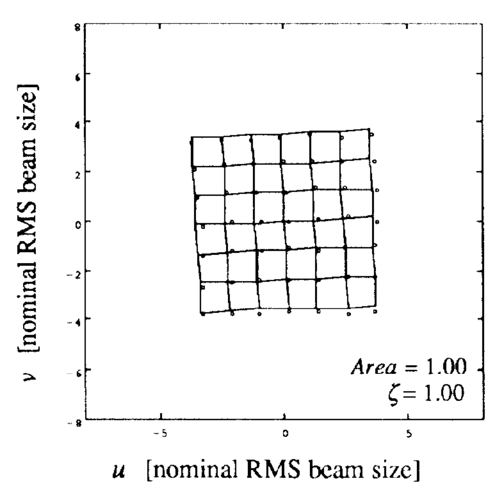

Fig 2. Normalized phase space grid for an RTL BPM pair. Points are BPM data to which the grid is fitted, and linearity is clear.

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The area of the grid is calculated (section III) and normalized to the area at BPM-0. With no acceleration or X-Y coupling, the case for this data, the area should remain constant and the grid will rotate but remain square as different BPM pairs are selected down the beam line. A focusing error in the line will linearly distort the grid from its initially square orientation with no area change (quantified in section III).

An example of a geometric aberration which is clearly distinguished is shown in Fig 3. The data is from a horizontal grid scan across the damping ring extraction septum.

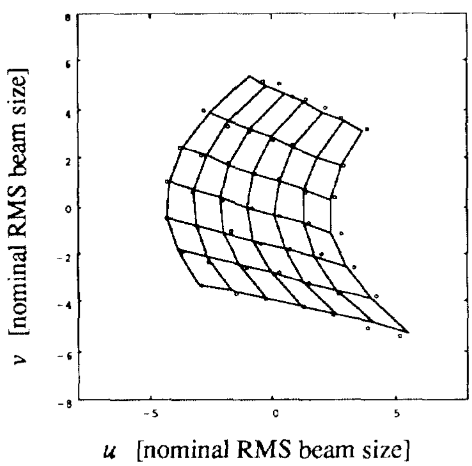

Fig 3. Horizontal grid scan across the damping ring extraction septum showing geometric aberration (fitted to second order).

III. CONCEPT

If we start with the initial transformation from x_1, x_2 to x_1, x_1' ($=dx/ds$) already applied and begin in these phase space coordinates, then the normalized phase space transformation is straight forward, with the exception of the initial grid 'squaring' at BPM-0. Normalized phase space at BPM-0 is obtained as in (1) by the transformation

$$\vec{u}_0 = \left(\sqrt{\varepsilon}\mathbf{A}_0\right)^{-1}\vec{x}_0 \; ; \quad \mathbf{A}_0 \equiv \frac{1}{\sqrt{\beta_0}}\begin{bmatrix} \beta_0 & 0 \\ -\alpha_0 & 1 \end{bmatrix} ; \quad \vec{x}_0 \equiv \begin{bmatrix} x_0 \\ x_0' \end{bmatrix} . \tag{4}$$

For ease of graphic interpretation it is convenient to **choose** the Twiss parameters at BPM-0 in order to produce a square grid there. The Twiss parameters of the *beam* may also be used, however, depending on the correctors used, the grid will then not be square initially so that downstream optical distortions, when they occur, will be more difficult to discern.

The initial Twiss parameters are chosen by finding the transformation matrix $\mathbf{M}$, from x_0, x_0' which 'squares' the initial grid while maintaining its area. The left side of Fig 4 suggests the three equations

$$\mathbf{M}\vec{p}_0 = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} ; \quad \mathbf{M}\vec{q}_0 = a\begin{bmatrix} 0 \\ 1 \end{bmatrix} ; \quad \det(\mathbf{M}) = 1 . \tag{5}$$

Here a is a scale factor adjusted so that the area is unchanged, and the vectors p_0 and q_0 (Fig 4) are extracted from the data at BPM-0. The transformation $\mathbf{M}$ is calculated as

$$\mathbf{M} = \sqrt{\left|\det(\mathbf{G}_0)\right|}\,\mathbf{G}_0^{-1} ; \quad \mathbf{G}_0 \equiv \left[\vec{p}_0 \;\; \vec{q}_0\right] = \begin{bmatrix} p_1 & q_1 \\ p_2 & q_2 \end{bmatrix} . \tag{6}$$

The grid area S_0 is calculated as the cross product magnitude

$$S_0 = \left|\vec{p}_0 \times \vec{q}_0\right| = \left|\det(\mathbf{G}_0)\right| . \tag{7}$$

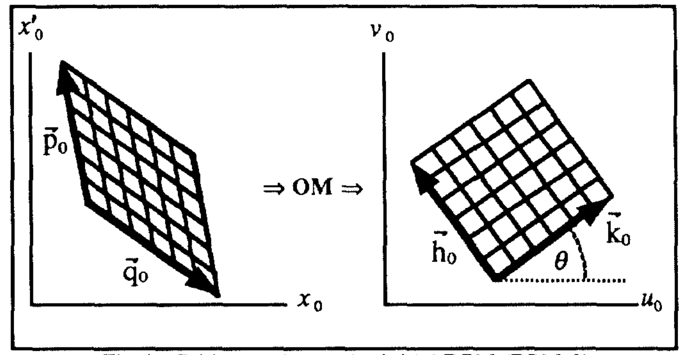

Fig 4. Grid *squaring* at the initial BPM (BPM-0)

Substituting $\mathbf{M}$ for $\mathbf{A}_0$ into (4) provides a square grid with unchanged area (except for the factor of ε) at BPM-0. However, $\mathbf{M}$ is not necessarily lower triangular and therefore identification of the elements of $\mathbf{M}$ as the chosen Twiss parameters is difficult. To convert $\mathbf{M}$ into a lower triangular matrix, we simply rotate it through the angle θ (Fig 4) with the orthogonal matrix $\mathbf{O}$ (equivalent to defining the initial betatron phase advance).

$$\mathbf{OM} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\mathbf{M} = \begin{bmatrix} \beta_0^{-\frac{1}{2}} & 0 \\ \alpha_0\beta_0^{-\frac{1}{2}} & \beta_0^{\frac{1}{2}} \end{bmatrix} \tag{8}$$

The angle θ is solved using the lower triangular condition $(\mathbf{OM})_{12} = 0$, and the initial Twiss parameters follow.

$$\tan\theta = -\frac{\mathbf{M}_{12}}{\mathbf{M}_{22}} ; \quad \beta_0 = \left(\mathbf{OM}\right)_{11}^{-2} ; \quad \alpha_0 = \frac{\left(\mathbf{OM}\right)_{21}}{\left(\mathbf{OM}\right)_{11}} \tag{9}$$

Propagating these Twiss parameters to the other BPMs then reduces to application of (2). The transformation to normalized phase space at a downstream BPM pair follows as in (4) with subscript 0 replaced by 1. Now express phase space at BPM-1 in terms of that at BPM-0 as

$$\vec{x}_1 = \mathbf{R}^{(0:1)}\vec{x}_0 \; , \tag{10}$$

and decompose the design transfer matrix $\mathbf{R}$ into initial and final design Twiss parameters and a rotation matrix Ψ (betatron phase advance).

$$\mathbf{R}^{(0:1)} = \mathbf{A}_1\Psi_{01}\mathbf{A}_0^{-1} \tag{11}$$

The design Twiss matrices $\mathbf{A}_1$ and $\mathbf{A}_0$ are as defined in (4), and $\Delta\psi_{01}$, implicit in Ψ_{01}, is the design betatron phase advance from BPM-0 to BPM-1.

A focusing error between BPM-0 and BPM-1, will change the final Twiss parameters and the phase advance between these points. The effect of a focusing error on the transfer matrix is written as

$$\widehat{\mathbf{R}}^{(0:1)} = \widehat{\mathbf{A}}_1 \widehat{\Psi}_{01} \mathbf{A}_0^{-1} , \quad (12)$$

where the 'hatted' (perturbed) matrices are introduced to distinguish them from the design. Combining (4), (10), and (12)

$$\vec{u}_1 = \left(\sqrt{\varepsilon}\mathbf{A}_1\right)^{-1} \widehat{\mathbf{A}}_1 \widehat{\Psi}_{01} \mathbf{A}_0^{-1} \vec{x}_0 = \left(\mathbf{A}_1^{-1} \widehat{\mathbf{A}}_1\right) \widehat{\Psi}_{01} \vec{u}_0 . \quad (13)$$

From (13) it is clear that with no focusing errors ($\hat{\mathbf{A}}_1 = \mathbf{A}_1$) the grid at BPM-1 is simply a clockwise rotation of the grid at BPM-0 through $\Delta\psi_{01} > 0$. Furthermore, since the determinants of all matrices in (13) are unity, the grid area is invariant and independent of the choice of initial Twiss parameters.

The quality of the beta match is indicated in the 'squareness' of the grid. Much like a matched beam remaining circular in normalized phase space, the grid should remain square for a perfect lattice. The 'squareness' of each grid is defined as the sum of the squares of the lengths of the grid spanning vectors h_1 and k_1 at BPM-1 (as is shown for BPM-0 in Fig 4) which are taken from the grid fit to the data. Writing the grid as a 2×2 matrix of these two vectors, the 'squareness' can then be conveniently written as the trace of $\mathbf{H}_1$ times itself transposed.

$$\mathbf{H}_1 \equiv \left[\vec{h}_1 \ \vec{k}_1 \right] , \quad (14)$$

$$\mathrm{tr}(\mathbf{H}_1 \mathbf{H}_1^T) = \left|\vec{h}_1\right|^2 + \left|\vec{k}_1\right|^2 \quad (15)$$

Since h_0, k_0 are simply two identically scaled orthonormal vectors rotated through θ, $\mathbf{H}_0$ can be defined as

$$\mathbf{H}_0 \equiv \left[\vec{h}_0 \ \vec{k}_0 \right] = \sqrt{\left|\det(\mathbf{G}_0)\right|}\ \mathbf{O} . \quad (16)$$

Transformation from $\mathbf{H}_0$ to $\mathbf{H}_1$ is the same as transformation from u_0 to u_1 in (13)

$$\mathbf{H}_1 = \left(\mathbf{A}_1^{-1} \widehat{\mathbf{A}}_1\right) \widehat{\Psi}_{01} \mathbf{H}_0 \equiv \mathbf{F}_1 \mathbf{H}_0 , \quad (17)$$

where the definition of $\mathbf{F}_1$ is here implicit. Rewriting (15) using (16) and (17) gives the 'squareness' in terms of $\mathbf{F}_1$.

$$\mathrm{tr}\left(\mathbf{H}_1 \mathbf{H}_1^T\right) = \mathrm{tr}\left(\mathbf{F}_1 \mathbf{H}_0 \mathbf{H}_0^T \mathbf{F}_1^T\right) = \left|\det(\mathbf{G}_0)\right| \mathrm{tr}\left(\mathbf{F}_1 \mathbf{F}_1^T\right) \quad (18)$$

Substituting $\mathbf{A}_1$ and its perturbed counterpart into the definition of $\mathbf{F}_1$ produces a beta match parameter, ζ, in terms of the Twiss parameters at BPM-1 of the current pair, which is invariant until a second focusing error is encountered.

$$\zeta \equiv \frac{1}{2}\mathrm{tr}\left(\mathbf{F}_1 \mathbf{F}_1^T\right) = \frac{1}{2}\mathrm{tr}\left\{\left(\mathbf{A}_1^{-1}\widehat{\mathbf{A}}_1\right)\widehat{\Psi}_{01}\widehat{\Psi}_{01}^T\left(\mathbf{A}_1^{-1}\widehat{\mathbf{A}}_1\right)^T\right\} =$$
$$\frac{1}{2}\mathrm{tr}\left\{\left(\mathbf{A}_1 \mathbf{A}_1^T\right)^{-1}\left(\widehat{\mathbf{A}}_1 \widehat{\mathbf{A}}_1^T\right)\right\} = \frac{1}{2}\mathrm{tr}\left(\sigma_1^{-1}\widehat{\sigma}_1\right) =$$
$$\frac{1}{2}\left\{\frac{\beta_1}{\widehat{\beta}_1} + \frac{\widehat{\beta}_1}{\beta_1} + \beta_1\widehat{\beta}_1\left(\frac{\alpha_1}{\beta_1} - \frac{\widehat{\alpha}_1}{\widehat{\beta}_1}\right)^2\right\} \quad (19)$$

The 1/2 is introduced so the matched case will result in $\zeta = 1$, and the beam matrix, σ, is introduced, which follows from

$$\mathbf{A}\mathbf{A}^T = \begin{bmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{bmatrix} \equiv \frac{\sigma}{\varepsilon} . \quad (20)$$

The 'hatted' quantities in (19) represent the perturbed beam parameters, while those without the 'hats' represent the design. From (15), (18), and (19), the beta beat amplitude, ζ, at BPM-1 of each BPM pair (indicated at the bottom of the grid in Fig 2), is a direct result of the fitted grid vectors h_1 and k_1.

$$\zeta = \frac{1}{2}\frac{\left|\vec{h}_1\right|^2 + \left|\vec{k}_1\right|^2}{\left|\det(\mathbf{G}_0)\right|} \quad (21)$$

A short proof of the invariance of ζ is shown by calculating ζ_i at a point farther downstream, if we transport both the design beam, σ, and the perturbed beam, $\widehat{\sigma}$, to point-i through the same design $\mathbf{R}$ and use common matrix properties.

$$\zeta_i = \frac{1}{2}\mathrm{tr}\left\{\left(\mathbf{R}\sigma\mathbf{R}^T\right)^{-1}\left(\mathbf{R}\widehat{\sigma}\mathbf{R}^T\right)\right\} =$$
$$\frac{1}{2}\mathrm{tr}\left\{\left(\mathbf{R}^{-1}\mathbf{R}\right)^T \sigma^{-1}\left(\mathbf{R}^{-1}\mathbf{R}\right)\widehat{\sigma}\right\} = \frac{1}{2}\mathrm{tr}\left\{\sigma^{-1}\widehat{\sigma}\right\} = \zeta \quad (22)$$

As a final point, note that although the choice of initial Twiss parameters at BPM-0 does not affect the invariance of ζ or of grid area, this choice does however affect the step size of ζ when a focusing error is encountered. Therefore, the Twiss parameters of the beam itself should be used at BPM-0 (rather than the grid squaring parameters used in Fig 2) if a directly qualitative beta matching study is of prime interest.

IV. CONCLUSIONS

The grid technique has advantages over simply comparing a measured oscillation with a model. The oscillation comparison does not assume an area preserving map and, for example, may confuse a BPM calibration error with a focusing error. Furthermore, since the grid scan does not force linearity, the technique will reveal nonlinearities over the scan range and therefore is useful for qualitatively evaluating geometric effects.

Application to the SLC RTL beam line shows only some small focusing errors along the RTL with no measurable geometric aberrations downstream of the extraction septum. This scheme helped eliminate one potential source for a suspected RMS emittance growth within the RTL itself.

V. ACKNOWLEDGEMENTS

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VI. REFERENCES

[1] M.D. Woodley,*et.al.*, "Control of Machine Functions or Transport Systems", SLAC-PUB-3086, March 1983