A Simple Analytic Estimate of the Loss Parameter of a Large Tapered Chamber *

James J. Welch, Laboratory of Nuclear Studies, Cornell University, Ithaca, NY 14853

Abstract

When bunch lengths are much smaller than the dimensions of the structure of interest, it is often either impossible or excessively time consuming to obtain estimates of the loss parameter by numerical methods. Furthermore, numerical results are not generally scalable to other bunch lengths, chamber angles or sizes. In the case of a cylindrical chamber very large compared with the bunch length, and tapers on each end, an estimate for the loss parameter may be made by first making some assumptions about the distribution of electromagnetic fields, and then calculating the electromagnetic energy left behind when the bunch leaves the chamber. The result is an analytic formula for the loss parameter, with no free or empirical parameters, that agrees well with TBCI calculations of a 20° tapered chamber. Besides its intuitive appeal, this formula is especially useful in providing design assistance as to the best taper angle, diameter, and chamber lengths.

Electromagnetic Field Model

It is well known that for $\gamma >> 1$ in free space or in a uniform beam tube, the electromagnetic field of a short bunch of electrons is greatly compressed, so that it is almost entirely perpendicular to the direction of motion. The energy stored in the electromagnetic field traveling with the bunch is likewise confined to a thin pancake whose thickness is approximately the bunch length. If we allow the cross section of the beam tube to have a sudden change, the pancake can deform, generate reflections or even thicken, but it must do so within the constraints imposed by the finite speed of light, i.e. causality.

The general vacuum chamber geometry considered here is shown in figure 1. Under certain conditions, the loss parameter is independent of the outer radius R and may be calculated as if the taper continued indefinitely. For a highly relativistic uniform bunch of length Δl, if the path length **ABC** is greater than L by at least Δl, then the perturbation of the electromagnetic fields at **A** cannot reflect off the outer wall at **B** and catch up with the back of the bunch at **C**, where the bunch leaves the chamber. This is only approximately true for a gaussian bunch. Once the

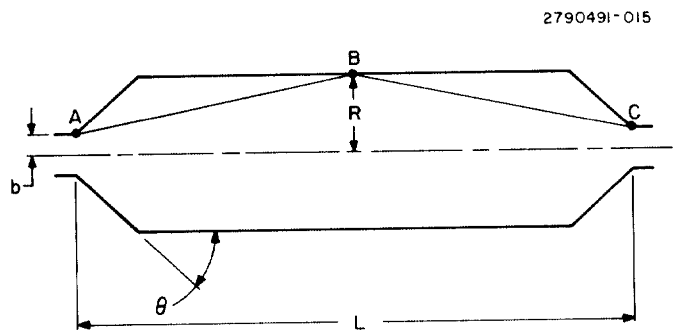

Figure 1: A cylindrically symmetric chamber with tapered ends is shown in this figure. Points **A** and **B** are the beginning and end of the chamber. Point **B** is the point where a signal from **A** could reflect and reach **C** in the minimum time.

bunch re-enters the beam pipe, it is difficult for reflections of the electromagnetic pancake to catch up to the back of the bunch because the group velocity in the beam pipe is substantially less than c, and the walls of the beam pipe have nonzero resistivity. So any energy left in the chamber after the bunch leaves cannot effectively interact with the bunch charge and will be 'lost'. From the geometry in figure 1, it can be seen that the energy lost is approximately independent of the outer radius R if

$$\sqrt{2(R-b)^2+L^2}-L>\sigma \tag{1}$$

Bench measurements showing this effect does indeed occur can be found in [1].

When a short uniformly charged bunch of length Δl and charge ΔQ first enters the taper, we can expect on physical grounds, that the electromagnetic field will look like that shown in figure 2. The shape of the leading edge of the shell is determined by causality. It is a constant distance from point **A** in the section shown in figure 2. For radial coordinate r less than b, the field lines are straight; essentially the same as when the bunch was in the beam pipe. Causality would allow the trailing edge to be further behind the leading edge than Δl shown in the figure. This thickening of the shell could come about from perturbed fields generated at different azimuthal points but at the same longitudinal position as point **A** in figure 2. The effect would be most pronounced for $L < b^2/\sigma$ and stronger

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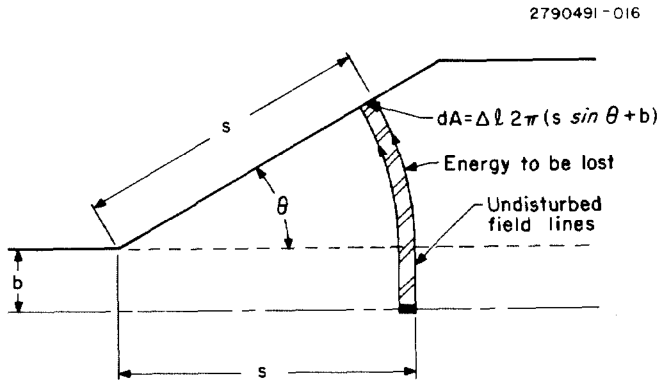

Figure 2: A highly relativistic uniformly distributed bunch of electrons will generate a thin shell of intense electromagnetic field in a shape approximately as shown in this figure.

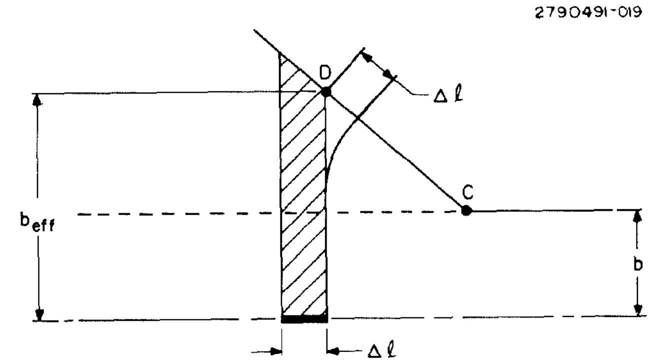

Figure 3: The effective exit radius of the beampipe b_{eff} is defined in this figure as the radius for which a signal from point **D** could catch up with the back of the bunch at point **C**.

for steeper taper angles. We will neglect this effect. By neglecting it, we will overestimate the strength of the electric field in the shell as well as the energy loss.

We will further assume that reflections from the entrance taper, traveling in the opposite direction of the bunch, are not significant. They are not appreciably seen in TBCI calculations, and in principle, could be removed by smoothing the discontinuity at point **A**. If there are no reflections, there should be no field behind the segment of charge.

By applying Gauss's law to the surface bounded by the 'wavefronts' associated with the front and back of the bunch and the outer annular segment of length Δl located at angle θ we have for $r \geq b$,

$$E = \frac{2\Delta Q}{\Delta l(s\sin\theta + b)} \tag{2}$$

The magnetic field can be computed in exactly the same manner using Ampere's law. The total electromagnetic energy in the field volume $r \geq b$ is,

$$\Delta U = \frac{2}{8\pi}\int_0^\theta E^2 dV = \frac{2(\Delta Q)^2}{\Delta l}\int_0^\theta \frac{d\theta}{(s\sin\theta + b)} \tag{3}$$

A factor of two was included to take into account the magnetic field contribution. The integral can be evaluated for two separate cases, $s > b$ and $s < b$. We are mainly interested in $s > b$ where,

$$\Delta U = \frac{(\Delta Q)^2}{\Delta s}\frac{2s}{\sqrt{s^2-b^2}}\ln\left(\frac{(s+\sqrt{s^2-b^2})\tan\theta/2 + b}{(s-\sqrt{s^2-b^2})\tan\theta/2 + b}\right) \tag{4}$$

A simpler formula can be obtained by making slightly different assumptions about the shape of the electric field lines. When s is large compared with the beam pipe radius b, the 'wavefront' approximately resembles a pure spherical shell with radius s centered on the chamber axis, extending to angle θ. The energy stored in fields for $r > b$ when the particle is at position $s = L$, is calculated by integrating the energy from a minimum angle θ_0, where

$$\theta_0 = \tan^{-1}(b/L) \tag{5}$$

to the taper angle θ. The result of this integration for uniformly charged bunches is,

$$U = \frac{2Q^2}{\Delta l}\ln\left[\frac{\tan(\theta/2)}{\tan(\theta_0/2)}\right] \tag{6}$$

Equation 4 reduces to equation 6 when $b/L << 1$.

Formulae for the Loss Parameter

Notice the field energy stored in the volume $r > b$ grows with increasing s. We will assume that when the beam reaches the far end of the chamber, the field is reestablished in the beam pipe and no further energy is lost. That is, when the bunch position is at $s = L$, the energy in the electromagnetic field for $r \geq b$ is 'scraped off' and can never again interact with the bunch. This assumption is similar to one made by Dôme [3] that for $r < b$ the fields of a point charge in the cavity are the same as they were when the charge was in the beampipe. An analogous argument was made by V.E. Balakin and A.V. Novokhatsky [2] for energy lost by a structure consisting of a sudden reduction in pipe diameter.

Of course, an assumption of pure scraping for $r < b$ precludes diffraction of fields at the exit end of the chamber which would tend not to scrape off so much field energy. Diffraction effects at the exit of the chamber occur if reflections of the front of the pulse from the exit taper (point **D** in figure 3) can reach the beampipe opening before the back of the bunch. A shallow taper will tend to capture much more of the electromagnetic energy. If we replace Δl in figure 3 with σ, then the effective exit radius for gaussian bunches may be defined as,

$$b_{eff} \equiv b + \sigma\cot\theta/2 \tag{7}$$

Taking the formulae for the energy loss of a short uniform bunch, equations 4 and 6, and assuming no thickening of the electromagnetic pulse, we can average over a gaussian longitudinal distribution to arrive at an expression for the energy loss for a gaussian bunch.

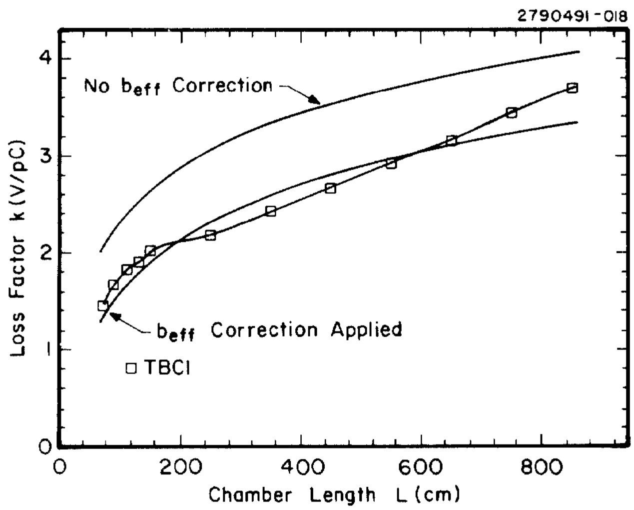

Figure 4: TBCI calculations for a 20 degree tapered chamber with beam pipe radius of 2.5 *cm* are plotted with the equation 6, with and without the modification due to diffraction applied.

Let the charge per unit length be λ and substitute λ for $\Delta Q/\Delta l$ into either equation 4 or 6. The expressions for the energy lost can then be written in the form,

$$dU = \lambda^2 f(L,\theta,b)ds \tag{8}$$

where ds is an infinitesimal length of charge. For gaussian bunches,

$$\lambda = \frac{Q}{\sqrt{2\pi}\sigma}e^{-s^2/2\sigma^2} \tag{9}$$

Integrating dU yields,

$$U = \frac{1}{2\sqrt{\pi}}\frac{Q^2}{\sigma}f(l,\theta,b) \tag{10}$$

The loss parameter is defined as $k = U_{loss}/Q^2$ where Q is the total bunch charge and U is the total bunch energy loss. Applying this form to equation 6 yields for a gaussian bunch,

$$k = \frac{1}{\sigma\sqrt{\pi}}\ln\left[\frac{\tan(\theta/2)}{\tan(\theta_0/2)}\right] \tag{11}$$

where all variables are assumed to be written in CGS units. In more commonly used units:

$$k\ [V/pC] = \frac{0.508}{\sigma\ [cm]}\ln\left[\frac{\tan(\theta/2)}{\tan(\theta_0/2)}\right] \tag{12}$$

Comparison with other Estimates

A comparison between TBCI calculations [5] and the analytic formula, equation 6, for 20° tapers is shown in figure 4. With no diffractive correction applied, i.e., $\theta_0 = \tan^{-1}(b/L)$, equation 6 consistently gives a higher estimate of the loss factor, particularly for shorter chamber lengths. Rather surprising is the fact that the difference between equation 6 and the TBCI results is more or less constant with respect to chamber length, indicating the corresponding difference in energy loss occurs only when the bunch enters or leaves the chamber. Figure 4 also shows that, even though for most of the range of chamber lengths calculated, the 'catch up' condition (equation 1) is not obtained, the effects of the reflections are small. When a diffractive correction $\theta_0 = \tan^{-1}(b_{eff}/L)$ is applied, there is constant reduction of the loss of approximately the right magnitude to account for the discrepancy between the TBCI data and the formula.

The other comparison I will make is with an analytic formula derived for 90 degree tapers by Heifets [4]. That is,

$$k = \frac{1}{\sigma\sqrt{\pi}}\ln\frac{\sigma L}{b^2} \tag{13}$$

subject to the restriction $L >> b^2/\sigma$. When $\theta = 90°$ is put into equation 6 and $L >> b$ the difference in k between and equation 6 and equation 13 approaches

$$\frac{1}{\sigma\sqrt{\pi}}\ln\frac{2b}{\sigma} \tag{14}$$

This difference is significant compared with the total loss factor unless $L/b >> b/\sigma$, but this is exactly the limitation given to equation 13.

References

[1] J.N. Weaver, P.B. Wilson, and J.B. Styles, *Bench Measurements of Loss Impedance for PEP Beam Line Components,* SLAC-PUB-2284, March 1979,

[2] V.E. Balakin and A.V. Novokhatsky, Proc. of the 12th Intern. Conf. on High Energy Accel. Fermilab, August 11-16, 1983, p117

[3] G. Dôme, *Wake Potentials of a Relativsitic Point Charge Crossing a Beam-Pipe: An Analytical Approximation* IEEE Trans. Nucl. Sci., vol. NS-32, No. 5, October 1985

[4] S. A. Heifets, *Diffractive model of the high-frequency impedance*, Phys. Rev. D, Vol. 40, No. 9, p 3097, Nov. 1989

[5] The TBCI calculations were performed by C. Chen, and will soon be appearing in a CLNS publication available from Newman Laboratory of Nuclear Science, Cornell University.