A New Wake-Potential Calculation Method Using Orthogonal Polynomials*

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Abstract

We present, in this paper, a new method for calculating the wake potential of a bunched beam of arbitrary charge distribution, directly from the wake potential of a shorter bunch, by using orthogonal polynomial expansions. Combined with the table-look-up technique, this method leads to an effective computation scheme for repetitive evaluation of wake potentials of different charge distributions under the same boundary conditions.

I. INTRODUCTION

Theoretically, the wake potential of a bunched beam of arbitrary charge distribution can be calculated by using the wake function of a point charge as a Green function [1]. In reality, except for a few very simplified geometries, the wake function of a point charge is impossible to obtain analytically. More difficulties emerge when one tries to compute the wake function (delta function wake) numerically, because of the singularity in the time domain and the infinite number of resonances in the frequency domain. Although several approximations in the time domain have been proposed [2,3,4], it is not clear that these approximations can give satisfactory results for all cases.

The difficulty of evaluating the wake function of a point charge is avoided by computing the wake potential of a nonsingular charge distribution of extended dimension. A number of computer programs have been developed for calculating the wake potentials of charged-particle bunches in various geometries [5,6,7]. Nonetheless, even with the most advanced computers, a wake potential calculation still requires a significant amount of computer time. Hence, it is not practical to use these programs to calculate wake potentials repeatedly in a simulation program for beam stability or beam-beam interaction in accelerators. A conventional method for a fast computation is to calculate the effective impedance in the frequency domain from the resonant modes of a structure, and then to Fourier-transform the results to the time domain [8]. Clearly, to use this method, one must know the modes up to very high frequencies. It is important to know the impedance of a structure before it is built; therefore one has to depend on the results from computations. In attempting to calculate the impedance numerically, one faces the same difficulties mentioned earlier. Trying to "unfold" the effective impedance of a bunch of finite length does not work in practice because of the extremely large weight factors where the effective impedance is small; e.g., for a Gaussian distribution, one would have to multiply the effective impedance with the weight factor $\exp(+\omega^2\sigma^2)$. Thus, a better scheme for rapid computation of wake potentials is required.

As will be discussed below, orthogonal polynomial expansions enable the wake potential of a bunch with arbitrary distribution to be calculated directly from a known wake potential of a short bunch. This method is similar to the Green function method; for any specific geometry, one needs to compute the wake potential of a short bunch only once. The wake potentials of each term in the expansion then can be obtained by (numerical) integrations. One may construct tables from the results and use the "table-look-up" technique [8] to increase the computation speed.

II. EXPANSIONS OF WAKE POTENTIALS

It is known that the wake function of a point charge $G(x,x')$ can be used as a Green function to calculate the wake potential $W_F(x)$ of a bunch with a charge distribution $F(x)$. If the wake function is a function of the difference of x and x' only, as in the case of an infinitely long beam pipe with open boundary conditions, the relation among $W_F(x)$, $G(x,x')$ and $F(x)$ can be written as

$$W_F(x) = \int_{-\infty}^{\infty} G(x-x')F(x')dx' = \int_{-\infty}^{\infty} G(x')F(x-x')dx' \quad , \qquad (1)$$

where it is understood that $G(y) \equiv 0$ for $y < 0$. In this paper we shall limit our discussions to the case for which Eq.(1) holds. One can infer from Eq.(1) that if $W_g(x)$ is the wake potential of a known function $g(x)$, and if a charge distribution $F(y)$ can be expressed as a convolution of $g(x)$ and some function $f(x)$, i.e.,

$$F(y) = \int_{-\infty}^{\infty} f(x)g(y-x)dx \quad , \qquad (2)$$

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then one can calculate the wake potential $W_F(y)$ of the distribution $F(y)$ by using the relation

$$W_F(y) = \int_{-\infty}^{\infty} W_g(t) f(y-t) dt \quad , \tag{3}$$

where

$$W_g(t) = \int_{-\infty}^{\infty} G(x') g(t-x') dx' \quad . \tag{4}$$

Our study here will be focused on those cases in which $g(x)$ is a simple function and in which the solutions of Eq.(2) exist for a given $F(y)$.

For an arbitrary distribution $F(y)$, it is usually impossible to find a closed form for the solution, but one can expand it [9] into orthogonal polynomials $O_n(y)$, i.e.,

$$F(y) = C(y) \sum_{n=0}^{\infty} a_n O_n(y) = \sum_{n=0}^{\infty} a_n S_n(y) \quad , \tag{5}$$

where $S_n(y) = C(y) O_n(y)$. The computing of the wake potential $W_F(y)$ is then transformed into finding the wake potential due to the functions $S_n(y)$. To relate the right-hand side of Eq.(5) to the known function $g(x)$, we use the convolution theorem of Fourier transformation to express $S_n(y)$ as

$$S_n(y) = \int_{-\infty}^{\infty} Q_n(x) g(y-x) dx \quad , \tag{6}$$

where $Q_n(x)$ is given by

$$Q_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\tilde{S}_n(\omega)}{\tilde{g}(\omega)} e^{i\omega x} d\omega \quad , \tag{7}$$

where $\tilde{S}_n(\omega)$ and $\tilde{g}(\omega)$ are the Fourier transforms of $S_n(y)$ and $g(y)$, respectively. Substituting Eq.(6) into Eq.(5); using Eqs.(2), (3) and (4), we derive

$$W_F(y) = \sum_{n=0}^{\infty} a_n \int_{-\infty}^{\infty} Q_n(y-t) W_g(t) dt \quad . \tag{8}$$

III. Evaluation of Specific Cases

We now consider some specific cases, using the results described in the previous section.

A. g(y) is a Gaussian Distribution[10,11]

In this case, it is advantageous to expand the function $F(y)$ into Hermite polynomials, thus $C(y) = (\sqrt{2\pi}\sigma)^{-1} \exp[-y^2/(2\sigma^2)]$, $O_n(y) = H_n[y/(\sqrt{2}\sigma)]$ and $g(y) = (\sqrt{2\pi}\sigma_1)^{-1} \exp[-y^2/(2\sigma_1^2)]$. Performing the necessary Fourier transformations [11] leads to

$$Q_n(x) = \frac{1}{\sqrt{2\pi}\sigma_2} \Big(\frac{\sigma}{\sigma_2}\Big)^n H_n\Big(\frac{x}{\sqrt{2}\sigma_2}\Big) \exp\Big(-\frac{x^2}{2\sigma_2^2}\Big) \quad , \tag{9}$$

where $\sigma_2^2 = \sigma^2 - \sigma_1^2$. Substituting Eq. (9) into Eq. (8) yields the wake potential of the (arbitrary) distribution F:

$$W_F(x) = \frac{1}{\sqrt{2\pi}\sigma_2} \sum_{n=0}^{\infty} a_n \Big(\frac{\sigma}{\sigma_2}\Big)^n \times \int_{-\infty}^{\infty} W_g(t) H_n\Big(\frac{x-t}{\sqrt{2}\sigma_2}\Big) \exp\Big[-\frac{(x-t)^2}{2\sigma_2^2}\Big] dt \tag{10}$$

where

$$W_g(t) = \frac{1}{\sqrt{2\pi}\sigma_1} \int_{-\infty}^{\infty} G(x') \exp\Big[-\frac{(t-x')^2}{2\sigma_1^2}\Big] dx' \quad , \tag{11}$$

is the wake potential of a bunch having a Gaussian charge distribution of standard deviation $\sigma_1 < \sigma$. Eq.(10) has recently been applied to build the wake potential tables in a beam stability simulation program [12].

When $F(x)$ is a Gaussian function, then the expansion coefficients $a_n = 0$ (except when $n = 0$). Equation (10) then leads to the well-known result that the wake potential of the longer Gaussian bunch can be expressed as a superposition of the wake potential of the shorter Gaussian bunch, as [13]

$$W_F(x) = \frac{1}{\sqrt{2\pi}\sigma_2} \int_{-\infty}^{\infty} W_g(s) \exp\Big[-\frac{(s-x)^2}{2\sigma_2^2}\Big] ds \quad . \tag{12}$$

We note again that the expansion of wake potential in terms of Hermite polynomials has been used in the beam stability simulation in conjunction with some point-charge Green functions or impedances [8,14,15]. The method described here differs from previous ones in that it uses the known wake potential of a Gaussian bunch instead of the Green function or impedance (which are in general difficult to obtain).

B. Distributions F(y) and g(y) have finite extents.[11]

We consider a special distribution function,

$$g(x) = [\theta(x+l) - \theta(x-l)]/(2l) \quad , \tag{13}$$

where $\theta(x)$ is the Heaviside step function and l is the half length of the short bunch. When $F(y)$ has a finite extent of half length L, one can expand it in terms of orthogonal polynomials defined on a unit interval, i.e., $O_n(y) = O_n(y/L)$ for $|(y/L)| \leq 1$, and $O_n(y) = 0$ elsewhere. It can be shown, by applying the residue theorem, that [11]

$$Q_n(x) = l \frac{d}{dx} \Big\{ \sum_{m=0}^{M} S_n[(x/L) - (2m+1)(l/L)] - \sum_{k=0}^{K} S_n[(x/L) + (2k+1)(l/L)] \Big\} \quad , \tag{14}$$

where M is the largest integer for $(2M+1)l \leq (L+x)$, K is the largest integer for $(2K+1)l \leq (L-x)$, and $S_n(y/L) = C(y/L)O_n(y/L)$ is defined in the same range as that of $O_n(y/L)$.

As an example, we consider a case in which $C(y/L) = (y/L)^2 - 1$, $O_n(y/L)$ is the Legendre polynomial $P_n(y/L)$, and $(L/2) < l < (2L/3)$. Applying Eqs. (8) and (14), we find that

$$W_F(x) = \sum_{n=0}^{\infty} a_n\left(\frac{l}{L}\right) \int_{-\infty}^{\infty} W_g(t) B_n(x-t)dt \quad , \qquad (15)$$

where

$$B_n(x) = \begin{cases} (2+n)\left[x_1 P_n(x_1) + \theta(1+x_2)x_2 P_n(x_2)\right] \\ \quad -n\left[P_{n-1}(x_1) + \theta(1+x_2)P_{n-1}(x_2)\right] \ , \\ \quad \text{for} \quad x \geq (L-l) \ , \\ (2+n)\left[x_1 P_n(x_1) - x_3 P_n(x_3)\right] \\ \quad -n\left[P_{n-1}(x_1) - P_{n-1}(x_3)\right] \ , \\ \quad \text{for} \quad (l-L) \leq x \leq (L-l) \ , \\ -(2+n)\left[x_3 P_n(x_3) + \theta(1-x_4)x_4 P_n(x_4)\right] \\ \quad +n\left[P_{n-1}(x_3) + \theta(1-x_4)P_{n-1}(x_4)\right] \ , \\ \quad \text{for} \quad x \leq (l-L) \ , \end{cases} \qquad (16)$$

$x_1 = (x-l)/L$, $x_2 = (x-3l)/L$, $x_3 = (x+l)/L$, $x_4 = (x+3l)/L$, and

$$W_g(t) = \frac{1}{2l}\int_{-\infty}^{\infty} G(x')[\theta(t-x'+l) - \theta(t-x'-l)]dx' \quad (17)$$

is the wake potential due to the short pulse of "rectangular" distribution.

IV. CONCLUSIONS

For any boundary conditions for which one can obtain the wake potential of a short bunch, Eqs.(3)-(8) allow us to calculate the wake potential of a longer bunch with arbitrary charge distribution. Compared with other methods, the method presented here has the advantages of better accuracy and higher efficiency. One needs to use the time-consuming wake potential programs only once and to apply the results to any other distribution function. This feature is especially useful in a beam stability or beam-beam interaction simulation, because the computing speed can be increased by using this method in conjunction with the table-look-up technique.

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