Root–mean–square Emittance Analysis of a Multiple Beam System

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Abstract

A multiple beam system is a potentially important means of generating, accelerating, and transporting high–brightness beams. In this work, general expressions for the root–mean–square emittance of a multiple beam system are derived. Of practical interest are arrays of identical beamlets. The emittance for rectangular and circular arrays are analyzed and expressed in terms of quantities associated with the individual beamlet distribution and the array geometry.

I. INTRODUCTION

The concept of a multiple beam system shows promise for several applications. High current beams can be realized by combining several beamlets [1,2] that are individually produced, accelerated and transported. Also, several new cathode concepts for the production of high current beams with reduced modulation requirements are based on arrays of micro–sized electron sources [3,4]. An important concern in such multiple beam systems is the beam emittance. In this letter, a general expression for the root–mean–square (rms) emittance is unambiguously derived. It is found that the emittance can be expressed in terms of the individual beamlet distribution and the array geometry. The results are applied to rectangular and circular arrays.

II. RMS EMITTANCE FOR MANY BEAMS

The rms emittance in $x-x'$ trace space is defined as

$$\epsilon_x = k[<x^2><x'^2>-<xx'>^2]^{1/2} \tag{1}$$

where x' is the gradient of a particle trajectory given by $x'=dx/dz=p_x/p_z$. The value k=1, proposed by Lawson [5] to designate the rms emittance, will be used throughout this letter. The squared rms emittance (SRE) can be written from Eq. (1) in determinant form as

$$\epsilon^2 = \begin{vmatrix} <x^2> & <xx'> \\ <xx'> & <x'^2> \end{vmatrix} \tag{2}$$

were the subscript x on ϵ is omitted for brevity. Considering a multiple beamlet system, the averaging integral of any quantity φ can be expressed as the sum of the individual averaging integrals as

$$<\varphi> = \frac{\Sigma\int\varphi\rho_i dxdx'}{\Sigma\int\rho_i dxdx'} = \frac{\Sigma<\varphi>_i I_i}{\Sigma I_i} \tag{3}$$

where $I_i=\int\rho_i dxdx'$ and the summation is performed over the total number of beamlets, i=1 to N. Using Eq. (3) for each element in Eq. (2) results in the SRE for the multiple beamlet system as

$$\epsilon^2 = \frac{1}{(\Sigma I_i)^2}\begin{vmatrix} \Sigma<x^2>_i I_i & \Sigma<xx'>_i I_i \\ \Sigma<xx'>_i I_i & \Sigma<x'^2>_i I_i \end{vmatrix}. \tag{4}$$

We now consider a system of beamlets with identical distribution functions in trace space. As shown in Fig. 1, the quantities for each beamlet are evaluated within their own $X-X'$ coordinate system whereas the location of the beamlet centroid is expressed as $<x>_i-<x'>_i$ in the system coordinates. Since all the distribution functions are assumed to be the same, we note that $<\varphi>=\Sigma<\varphi>_i/N$ and Eq. (4) becomes

$$\epsilon^2 = \begin{vmatrix} <X^2> & <XX'> \\ <XX'> & <X'^2> \end{vmatrix} + \frac{1}{N}\begin{vmatrix} \Sigma<x>_i^2 & <XX'> \\ \Sigma<x>_i<x'>_i & <X'^2> \end{vmatrix}$$
$$+ \frac{1}{N}\begin{vmatrix} \Sigma<x'>_i^2 & <XX'> \\ \Sigma<x>_i<x'>_i & <X^2> \end{vmatrix}$$
$$+ \frac{1}{N^2}\begin{vmatrix} \Sigma<x>_i^2 & \Sigma<x>_i<x'>_i \\ \Sigma<x>_i<x'>_i & \Sigma<x'^2>_i \end{vmatrix}. \tag{5}$$

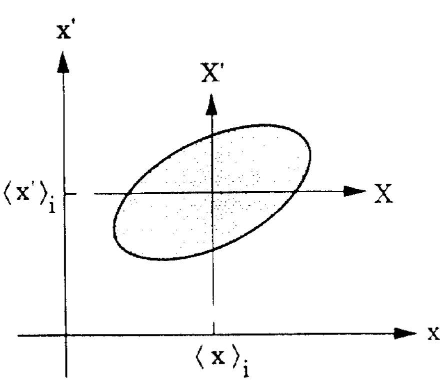

Figure 1. Trace–space representation of the i^{th} beamlet.

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It is noted that the first determinant is the SRE of the beam evaluated within its own coordinates X and X', and the second and third determinants are additional values due to a nonzero $<x>_i$ and $<x'>_i$. It is readily shown that, if the individual beam distribution is a point source, $\rho_0=\delta(X)\delta(X')$, the first three determinants are all zero and the last determinant may be recognized as the SRE of the same beam system, but with delta–function distributions for each beamlet.

It is important to note that Eq. (5) depends on the position of the system origin within the overall beam. Take for example the case when i=1; the obvious choice of $<x>=<x'>=0$ centers the beam on the system axis and causes all but the first determinant in Eq. (5) to vanish (See Fig. 1). A similar choice of $\Sigma<x>_i=\Sigma<x'>_i=0$ for the multiple beamlet system would make the expression for the SRE given by Eq. (5) unique. This condition will be assumed throughout the letter.

III. SRE FOR A RECTANGULAR ARRAY

We consider an m by n rectangular array of identical beamlets as shown in Fig. 2(a). For a simple analysis, it is assumed that the mean diverging angle of each beamlet is proportional to its mean position according to $<x'>_i=<x>_i/f$ (as if all the beamlets are diverging out from a focal point, $z=-f$, located upstream). The projection of beamlets onto $x-x'$ space is shown in Fig. 2(b). Note that the n rows of beamlets in $x-y$ space are superimposed in m distributions along the line $x'=x/f$ in $x-x'$ space. Thus, the terms involving $\Sigma<\varphi>_i/N$ in Eq. (5) are reduced to $\Sigma<\varphi>_j/m$ and the number of rows has no effect either on the trace–space quantities $<\varphi>$ or the total emittance of this array.

Substituting $<x'>_i=<x>_i/f$ into Eq. (5), the SRE of the rectangular array becomes

$$\epsilon^2 = \epsilon_0{}^2+<(X'-X/f)^2>G_r \tag{6}$$

where $\epsilon_0{}^2=<X^2><X'^2>-<XX'>^2$ is the SRE for each identical beamlet and G_r is a geometrical term given by

$$G_r = \frac{\Sigma<x>_i{}^2}{m} = \frac{a^2(m+1)}{3(m-1)}\,. \tag{7}$$

Note that the term $<(X'-X/f)^2>$, which is the mean square deviation from the dashed line, $X'=X/f$, in Fig. 2(b), is determined by both the focal length f and the beamlet distribution $\rho_0(X,X')$, whereas the geometrical term G_r is determined only by the array parameters. Note also that the geometrical term rapidly approaches $a^2/3$ as m becomes large. Thus, the SRE of the array is almost independent of m for large m. It can also be shown that for the m by n array where m is even, Eq. (7) remains valid.

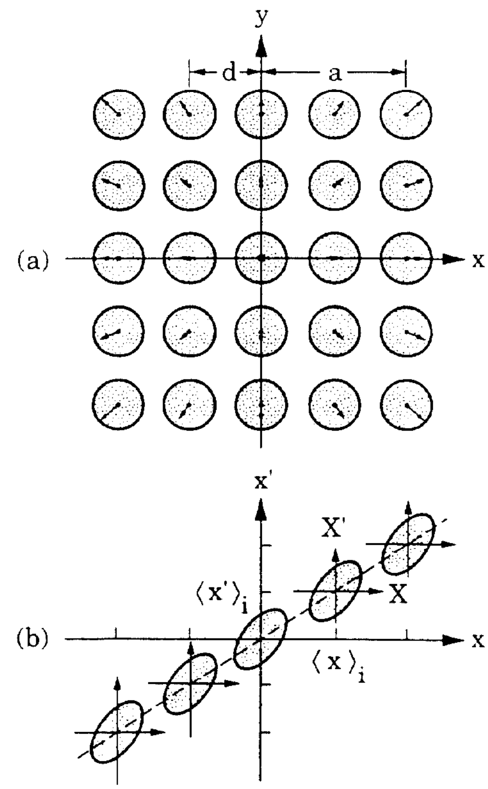

Figure 2. A rectangular array of identical beamlets. (a) An m×n (m=n=5) rectangular array is shown in x–y space. (b) The same array shown in x–x' trace space.

IV. SRE FOR A CIRCULAR ARRAY

We consider here a circular array in which N identical beamlets are distributed in m concentric circles. The jth circle has radius r_j and number of beamlets n_j that are evenly spaced around the circle as shown in Fig. 5. Inclusion of cylindrically symmetric average diverging angles ($r_j'=dr/dz$) in this analysis seems to be useful. The projection of the circular array onto $x-x'$ trace space, for which $\Sigma<x>_i=\Sigma<x'>_i=0$, is shown in Fig. 3(b). It should be apparent from Fig. 5 that for the kth beamlet in the jth circle, $<x>_{jk}=r_j\cos(2\pi k/n_j)$ and $<x'>_{jk}=r_j'\cos(2\pi k/n_j)$. Using these relations in Eq. (5), one finds

$$\epsilon^2 = \begin{vmatrix} <X^2> & <XX'> \\ <XX'> & <X'^2> \end{vmatrix} + \frac{1}{N}\begin{vmatrix} \Sigma\sigma_j r_j{}^2 & <XX'> \\ \Sigma\sigma_j r_j r_j' & <X'^2> \end{vmatrix}$$

$$+\frac{1}{N}\begin{vmatrix} \Sigma\sigma_j r_j' & <XX'> \\ \Sigma\sigma_j r_j r_j' & <X^2> \end{vmatrix}$$

$$+\frac{1}{N^2}\begin{vmatrix} \sigma_j r_j{}^2 & \Sigma\sigma_j r_j r_j' \\ \sigma_j r_j r_j' & \Sigma\sigma_j r_j'^2 \end{vmatrix} \tag{8}$$

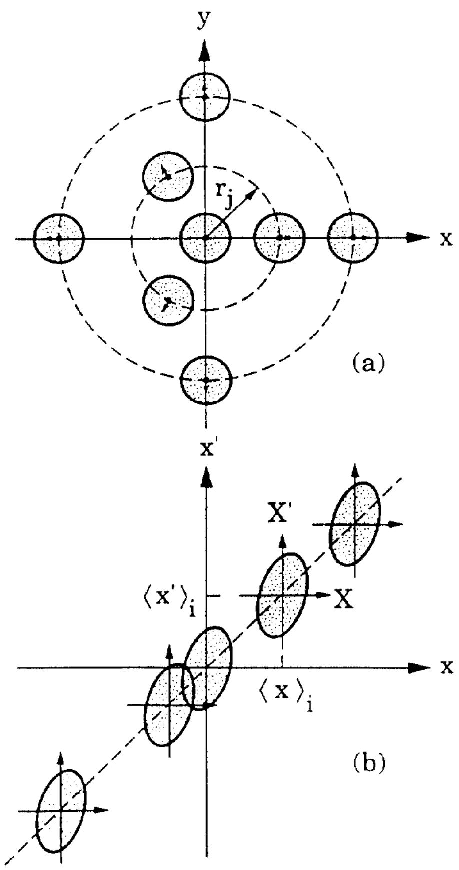

Figure 3. A circular array of identical beamlets. (a) Different number of beamlets are evenly distributed on each circle. The arrows indicate the divergence of each individual beamlet. (b) The same beam system is represented in x–x' space.

where $\sigma_j=\Sigma_k\cos^2(2\pi k/n_j)$. The first determinant is again ϵ_0^2, the SRE of an individual beamlet, and the second and third determinants are additional values contributed by the radii of circles and the diverging angles of beamlets. The last determinant is again the SRE of the same array geometry but with the original beamlets replaced by point sources of $\rho_0=\delta(X)\delta(X')$.

If we again assume an array with linearly diverging beamlets where the mean diverging angle of each beamlet is proportional to its radial position ($r_j'=r_j/f$), Eq. (8) can be reduced to

$$\epsilon^2 = \epsilon_0{}^2+<(X'-X/f)^2>G_c \qquad (9)$$

where $\epsilon_0{}^2=<X^2><X'^2>-<XX'>^2$ is the SRE for each identical beamlet and G_c is a geometrical term given by

$$G_c = \Sigma\sigma_j r_j{}^2/\Sigma n_j = \Sigma n_j r_j{}^2/2\Sigma n_j. \qquad (10)$$

It should be noted that the relation $\sigma_j=\Sigma_k\cos^2(2\pi k/n_j)=n_j/2$ has been used which holds for $n_j\geq 3$. However, since any beamlets placed at the center of the system have zero radius, their contribution to the geometrical term is zero, although they must still be counted in the denominator of Eq. (10). Thus, one can exclude the condition $n_j\geq 3$ for the beamlets placed at the center.

V. CONCLUSIONS

A general expression for the rms emittance of a multiple beam system has been derived. The evaluation of the rms emittance depends on the relative position of a beam with respect to its coordinate system. Thus, in order for the emittance to be uniquely defined, it is necessary to impose the condition that the centroid of the beam coincides with the origin of its coordinate system. The emittance for square and circular arrays of identical beamlets have been analyzed and expressed in terms of quantities associated with the individual beamlet distribution and the array geometry.

VI. REFERENCES

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