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Is Beta Modulation More or Less Potent than Tune Modulation?

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I. INTRODUCTION

Quadrupole power supply ripple, or synchrotron oscillations with non-zero chromaticities, not only causes the tunes to oscillate, but also drives time dependent beta function waves. Linear lattice modulation is well known to have profound effects on the dynamical response to nonlinear elements. Studies show that theory and simulation predict realistic sensitivities to high order proton beam-beam resonances only when modest levels of tune modulation are included in the model[1,2]. However, the effect of beta wave modulations in parametrically driving nonlinearities has not been well studied. Is "beta modulation" more or less potent than tune modulation in destroying stable resonance islands? We compare the response of fifth order resonance islands in the parameter space of modulation strength and period. The integrated picture that results incorporates piecemeal ideas from resonance trapping, sideband overlap, and Mathieu analysis [3-5]. In a frequency domain analysis, a resonance is characterized by two numbers - its order and its island tune, Q_I.

Study of the 2/5 resonance is motivated by the E-778 experimental setup in the Tevatron[6]. There, the quadratic variation of tune with amplitude (detuning) and the resonance driving are both second order in the strength of sixteen dominant sextupoles[7]. Here, one dimensional motion is studied in the lattice of Figure 1, so that a simple theoretical description is available with independent first order control over i) resonance strength, ii) detuning, iii) beta modulation amplitude depth, and iv) tune modulation depth.

* The fractional betatron tune is close to 2/5, so that fifth order islands are driven to first order in decapole strength, b_4.
* Three identical octupoles drive the detuning to first order in their strength, b_3. Their 60 degree spacing in betatron phase removes all first order contributions to phase space distortions.
* The two modulated quad pairs orthogonally drive a local beta error $\Delta\beta$ at the decapole, and shift the tune by ΔQ.

II. UNPERTURBED EQUATIONS OF MOTION

The effect of a thin decapole is written as

$$\Delta x' = -b_4 \beta^{5/2}_{dec} x^4 \qquad (1)$$

where x and x' are the normalized displacement and angle. Linear motion in (x,x') space is a circle. All beta functions are now implicitly set to nominal values of 1.0. It is convenient to go to action angle coordinates (J,ϕ), where

$$x = \sqrt{2J}\sin(\phi), \qquad x' = \sqrt{2J}\cos(\phi) \qquad (2)$$

Consider linear motion from reference point to decapole, nonlinear motion through it, and inverse linear motion back to the origin. The net motion is conveniently written as

$$H_p = \frac{1}{10\sqrt{2}} b_4 J^{5/2} [\sin(5(\phi+\phi_{dec})) - 5\sin(3(\phi+\phi_{dec})) + 10\sin(\phi+\phi_{dec})] \qquad (3)$$

Here ϕ_{dec} is the phase of the octupole relative to the reference point. The "discrete projection Hamiltonian", H_P, is shorthand for the *difference* equations of motion

$$\Delta\phi = \frac{\partial H_p}{\partial J}\Delta t, \qquad \Delta J = -\frac{\partial H_p}{\partial \phi}\Delta t \qquad (4)$$

where $\Delta t = 1$ is the discrete time step. These equations are only good to first order in b_4, unlike their counterparts in x,x' coordinates, that are valid for arbitrarily large b_4[4].

After including the linear motion and octupoles, 5 turn motion is represented by another discrete Hamiltonian,

$$H_5 = 2\pi(Q_0 - \tfrac{2}{5})J + H_3(oct) + H_5(dec) \qquad (5)$$

where the step in (4) is $\Delta t = 5$. The decapole part in (5) is

$$H_5(dec) = \frac{b_4}{50\sqrt{2}} J^{5/2} \sum_{n=0}^{4} \sin(5\theta) - 5\sin(3\theta) + 10\sin(\theta) \qquad (6)$$

and $\theta = \phi + \phi_{dec} + 2\pi n Q_0$. Applying the identity

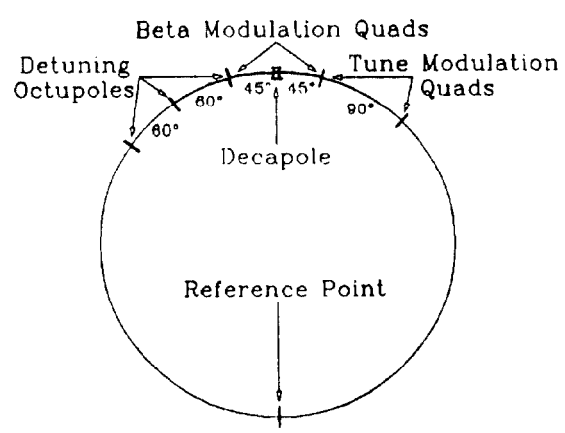

Figure 1 The model used for simulation and prediction.

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$$\sum_{n=0}^{N-1} \sin(A + nB) = \frac{\sin(NB/2)}{\sin(B/2)} \sin(A + \frac{N-1}{2} B) \quad (7)$$

to (6), with Q_0 close to 2/5, removes the first and third harmonics in ϕ. Finally, after a coordinate transformation

$$\psi = \phi + \phi_{dec} + 4\pi Q_0 \quad (8)$$

the discrete Hamiltonian takes a simple form

$$H_5 = 2\pi (Q_0 - \frac{2}{5}) J + \frac{9}{8} b_3 J^2 + \frac{b_4}{10\sqrt{2}} J^{5/2} \sin(5\psi) \quad (9)$$

that is valid to first order in decapole strength, b_4.

The net 5 turn motion described by (9) is small, as assumed in the application of (7), so H_5 can be thought of as a true Hamiltonian. That is, the *difference* equations (4) are well approximated by the classical *differential* equations

$$\frac{d\psi}{dt} = \frac{\partial H_5}{\partial J} = 2\pi(Q_0 - \frac{2}{5}) + \frac{9}{4} b_3 J + \frac{b_4}{4\sqrt{2}} J^{3/2} \cos(5\psi) \quad (10)$$

$$\frac{dJ}{dt} = -\frac{\partial H_5}{\partial \psi} = -\frac{b_4}{2\sqrt{2}} J^{5/2} \sin(5\psi) \quad (11)$$

Figure 2 shows the simulated motion with nominal parameters $Q_0 = 0.38$, $b_3 = 0.01$, and $b_4 = 0.0005$. The fixed points at the island centers are at an action J_0, given by

$$J_0 = \frac{8\pi}{9b_3}\left(\frac{2}{5} - Q_0\right) = 5.59 \quad (12)$$

plus a small term of first order in b_4. Differentiating (10) with respect to time, expanding relative to a fixed point using

$$J = J_0 + I \quad (13)$$

then dropping all terms higher than first in b_4 and assuming that $I << J_0$, leads to the representation of the motion by a single second order differential equation

$$\frac{d^2\psi}{dt^2} + \frac{(2\pi Q_I)^2}{5} \sin(5\psi) = 0 \quad (14)$$

The small oscillation "island tune" Q_I characterizing the resonance in the frequency (tune) domain analysis is given by

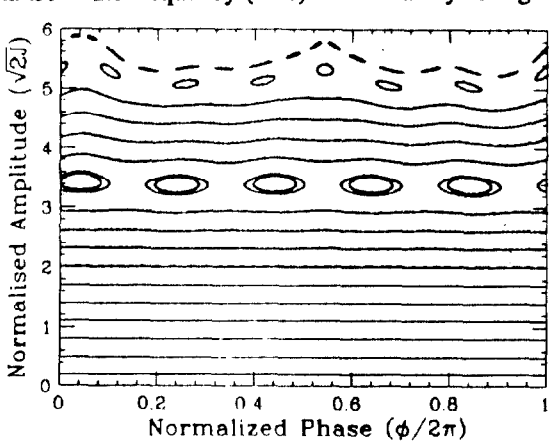

Figure 2 Phase space portrait showing a fifth order island chain, under nominal conditions.

$$(2\pi Q_I)^2 = \frac{45}{8\sqrt{2}} b_3 b_4 J_{fp}^{5/2} \quad (15)$$

Its value in the nominal case defined above is $Q_I = 0.0061$, typical of those found in the E-778 experiment.

III. PERTURBED EQUATIONS OF MOTION

Beta or tune modulation at a tune of Q_M is put into the master equations of motion (10) and (11) by substituting

$$b_4 \Rightarrow b_4 + \frac{5}{2}\Delta\beta \cos(2\pi Q_M t) \quad (16)$$

or

$$Q_0 \Rightarrow Q_0 + \Delta Q \cos(2\pi Q_M t) \quad (17)$$

Repeating the same procedure gives, for beta modulation,

$$\frac{d^2\psi}{dt^2} + \frac{(2\pi Q_I)^2}{5}[1 + \frac{5}{2}\Delta\beta \cos(2\pi Q_M t)] \sin(5\psi) = \frac{(2\pi Q_I)^2}{4(2/5 - Q_0)} \Delta\beta\, Q_M \cos(5\psi) \sin(2\pi Q_M t) \quad (18)$$

and for tune modulation

$$\frac{d^2\psi}{dt^2} + \frac{(2\pi Q_I)^2}{5}[1 + \frac{5}{2} \frac{\Delta Q}{(2/5 - Q_0)} \cos(2\pi Q_M t)] \sin(5\psi) = -4\pi^2 Q_M \Delta Q \sin(2\pi Q_M t) \quad (19)$$

Equations (18) and (19) differ from (14) in the same way - they both include parametric drive terms, on the left hand side of the equation, and direct drive terms, on the right hand side.

IV. SIMULATION RESULTS

Figure 3 compares simulation and prediction (symbols and lines) for the maximum stable beta modulation depth. The islands shrink as $\Delta\beta$ is slowly increased in the simulation, until they vanish. This scan is repeated for many integer modulation periods, T_M. The resonances appearing near

$$\frac{T_M}{T_I} = \frac{n}{2} \qquad \text{(integer } n,\ T_I = 1/Q_I) \quad (20)$$

are characteristic of the Mathieu problem of the stability of a parametrically driven harmonic oscillator, such as is obtained when the direct drive term is dropped from (18) and ψ is assumed to be small. The solid curves are the resonance boundaries found from a standard reference[8]. Agreement is good for the $n = 1$ and 3 resonances, but poor for $n = 2$, where the direct drive is important. Unphysically large depths $\Delta\beta > 0.1$ are generally required for instability.

Figure 4 compares simulated and predicted stability bounds for tune modulation. Comparison of (18) and (19) shows that the parametric strength $\Delta Q/(2/5 - Q_0)$ is analogous to $\Delta\beta$, so the vertical scales in Figures 3 and 4 represent the same range of parametric influence. Comparison also shows that tune modulation has a much stronger direct drive term. At the same parametric drive strength the direct drive is

$$(2(2/5 - Q_0) / Q_I)^2 = 42.9 \quad (21)$$

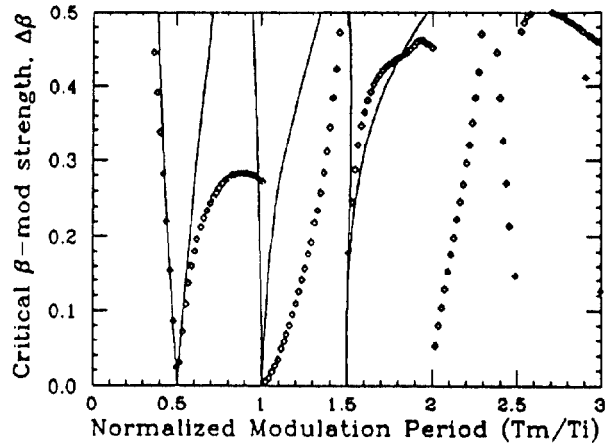

Figure 3 Maximum stable beta modulation depth versus modulation period, from simulation and prediction.

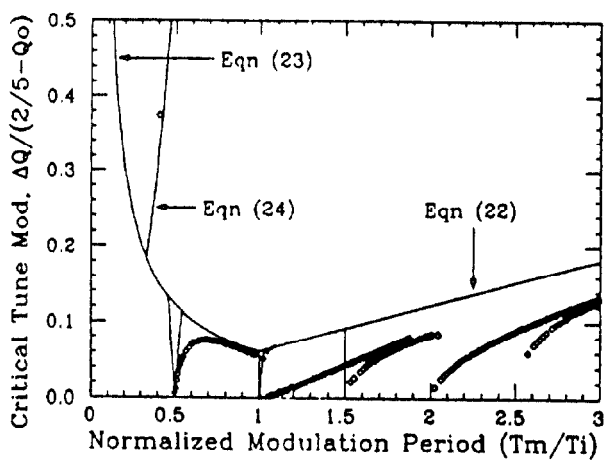

Figure 4 Maximum stable tune modulation depth versus modulation period, from simulation and prediction.

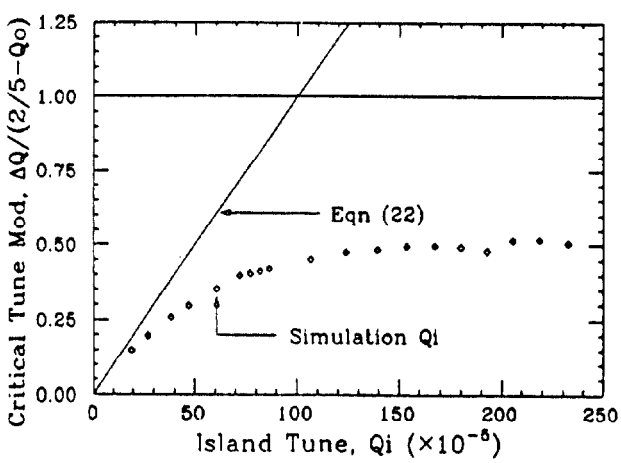

Figure 5 Tune modulation stability versus island tune Q_I, with a constant normalized adiabatic period $T_M/T_I = 10$.

times stronger with tune modulation. The extra lines in Figure 4, from a previous direct drive analysis[4], are

$$\left(\frac{\Delta Q}{Q_I}\right) = \frac{1}{5}\left(\frac{T_M}{T_I}\right) \tag{22}$$

and

$$\left(\frac{\Delta Q}{Q_I}\right)\left(\frac{T_M}{T_I}\right) = \frac{1}{5} \tag{23}$$

$$\left(\frac{\Delta Q}{Q_I}\right)^{1/4}\left(\frac{T_M}{T_I}\right)^{3/4} = \frac{4}{(5\pi)^{1/4}} \tag{24}$$

The first, (22), is the adiabatic limit for large T_M. The second, (23), accurately predicts stability between resonances 1 and 2. Above it, first order synchrobetatron sidebands overlap the fundamental. Motion is stable *above* the third line, (24), where many stable sidebands coexist peacefully.

Figure 5 shows that, with $T_M/T_I = 10$, the adiabatic boundary (22) is only valid when $\Delta Q << (2/5 - Q_0)$, the distance to the resonance. If $\Delta Q > (2/5 - Q_0)$, the resonance disappears when the zero amplitude tune adiabatically crosses 2/5. The suppression factor of 0.57 at the nominal Q_I indicated in Figure 5 improves the agreement with simulation data in Figure 4.

V. CONCLUSIONS

Modulation of a single quadrupole with an amplitude of $\Delta(kL)$ causes beta and tune modulations of order $\Delta\beta \approx \Delta(kL)$ and $\Delta Q \approx \Delta(kL)/4\pi$. If the nearest significant resonance is ΔQ_{res} away, (18) and (19) show that parametric drive with tune modulation is about $1/(4\pi\Delta Q_{res})$ times stronger than with beta modulation. In the nominal case here this factor is about 5, and it is unlikely to ever be less than 1. The direct drive term is $\Delta Q_{res}/\pi Q_I^2$ times stronger with tune modulation. This factor is nominally almost 200, and only in pathological situations does it approach unity. Direct drive dominates parametric drive in normal tune modulation conditions. On all counts, beta modulation is much less potent than tune modulation.

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VI. REFERENCES

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