# ON THE IMPEDANCE DUE TO SYNCHROTRON RADIATION*

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*Abstract*

A qualitative consideration of the impedance caused by the synchrotron radiation is given. The rigorous results such as the value of the threshold frequency and the maximum value of the impedance are obtained in a simple way.

## I. INTRODUCTION

The problem of the synchrotron radiation of a charge in a conductive vacuum chamber has been considered many times, see references in recent publications [1,2,3]. The rigorous consideration is based on the exact solution of the wave equation in a particular geometry (a charge moving between two conductive planes or in a toroidal chamber) and involves rather cumbersome calculations. Providing very useful reference models, these solutions call for a more simple heuristic picture of the physics involved which would clarify the situation, especially in cases where the exact solution is unknown.

Consider, for example, the results [2] for a charge moving along a circle with the radius R, in a pillbox cavity with the radius b and the height $h = 2g$. The real part of the impedance is given as a sum of $\delta$-functions due to the excitation of eigen modes of the cavity. The threshold frequency $\omega_{th}$ is much higher than the cutoff frequency $\omega_{cut} = \pi c/h$,

$$\omega_{th} \simeq \frac{c}{R}\left(\pi\,\frac{R}{h}\right)^{3/2} \gg 1\ ,$$

and the maximum value of the impedance

$$\left[\mathrm{Re}\,\frac{Z(n)}{n}\right]_{max} \simeq 300\,\frac{g}{R}\ \mathrm{Ohm}$$

is independent of b, see Appendix. This indicates that consideration based on the modal analysis is superfluous while the threshold frequency is a result of the intrinsic properties of the synchrotron radiation in a waveguide.

The threshold frequency may be obtained in the following way. As it is well known, the harmonic $n$ of the synchrotron radiation can be radiated only within the small angle $\theta$ with the plane of motion:

$$\theta < n^{-1/3}\ . \tag{1}$$

That follows from the intensity of the $n$-th harmonic of the synchrotron radiation [4]of an ultrarelativistic particle $\gamma >> 1$:

$$dW(n,\theta) = \frac{ce^2}{R}\,\frac{n^2}{6\pi^3} \times\left\{\epsilon^2 K^2_{2/3}\left(\frac{n}{3}\,\epsilon^{3/2}\right) + \epsilon\cos^2\theta\, K^2_{1/3}\left(\frac{n}{3}\,\epsilon^{3/2}\right)\right\}\,d\Omega \tag{2}$$

where $\epsilon^2 = 1/\gamma^2+\theta^2$. The intensity rolls off exponentially for harmonics

$$n > \frac{3}{2}\left(\frac{1}{\gamma^2}+\theta^2\right)^{-3/2} \tag{3}$$

in agreement with Eq. (1). The result Eq. (1) is the direct result of the Lorentz transformation of the dipole radiation in the moving frame of a particle and is an intrinsic feature of the synchrotron radiation.

Consider now a particle moving in $(x,y)$ plane between two conductive planes separated by the distance $h = 2g$. The radiated wave propagates between the planes as in a waveguide. Usually, this is possible if the wave frequency $\omega$ is above the cutoff frequency: $k = \omega/c > \pi/h$. For the waves with frequencies well above the cutoff frequency the propagation of the wave may be described in terms of the geometric optics with a wave vector $\vec{k}$, $|\vec{k}| = \omega/c = n\beta/R$. The boundary conditions on the conductive walls still require that the vertical component of the wave vector cannot be too small:

$$k_\perp = k\theta > \frac{\pi}{h}\ . \tag{4}$$

The cutoff frequency corresponds to the angle $\theta \simeq 1$. For the harmonics $n >> 1$ the angle is restricted by Eq. (1). Eqs. (1) and (4) give the threshold frequency:

$$n_{th} = \frac{\omega R}{c} = \sqrt{\frac{2}{3}}\left(\frac{\pi R}{h}\right)^{3/2}\ . \tag{5}$$

The synchrotron radiation with $n < n_{th}$ may be radiated only with the radiation angle $\theta > n^{-1/3}$, otherwise the boundary conditions cannot be satisfied. The probability of such radiation, as has been mentioned above, is exponentially small.

Hence, the single particle synchrotron radiation is exponentially small (see Eq. (2)) for harmonics $n < n_{th}$ and, as usually, with $n > n_{max} = \gamma^3$. Unfortunately, it is always $n_{th} << n_{max}$, and the decrease of the total radiated power due to suppression of the radiation with the harmonics $n < n_{th}$ is small.

The parameters defining the threshold frequency have to be clarified for more complicated structures such as a toroidal chamber, where there are several geometric dimensions (the height and the width of the chamber). With a good accuracy the polarization of the synchrotron radiation is such that the vector of the electric field is in the plane of motion (intensity of this polarization is 7/8 of the total intensity [4]). Therefore, only the height of the chamber enters in the boundary conditions for the tangential component of the electric field and in the threshold frequency.

It is worthwhile to consider the radiation length of the mode n. As usual, the radiation length or the length of the formation of the radiation can be defined as the length L

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where the phase of the radiation remains small: $|k_l L - \omega t| < \pi$. Using $k_l = (n/R)\cos\theta$, and Eq. (3) we obtain:

$$L = 2\pi R n^{-1/3} = 2\sqrt{\pi h R} . \quad (6)$$

The length L is small, $L << R$. Hence, the results obtained for a periodic motion on a circle with the radius R are valid also for an aperiodic motion or a periodic motion along a complicated trajectory with R being a local radius provided $R >> L \simeq \sqrt{hR}$.

It should be noted that the parameter $\omega_0 = c/R$ has a meaning of a fundamental frequency of oscillations. Thus, the same effect of suppression of the radiation below a threshold frequency can be expected for a dipole oscillating with frequency $\omega_0$ and propagating in a straight waveguide.

The effect of the finite conductivity can be estimated comparing the radiation length Eq. (6) with the absorption length of a wave. The absorption length $L_{abs}$ is defined as the length where the intensity of a wave decreases by a factor of $e$. It can be estimated from the definition of a $Q$-factor of a mode, $L_{abs} = ct = c/(Q\omega)$ with $Q = a/\delta$. Here $\delta$ is the skin depth, and $a$ is the geometric factor of the order of the beam pipe aperture. Clearly, the effect of finite conductivity is small provided $L_{abs} >> L$.

Let us consider now the coherent radiation of a bunch. The radiation of a bunch can be coherent if the bunch length is small compared with the wavelength. For a Gaussian bunch with rms length $\sigma$ that means $\omega\sigma/c < 1$ or $n < R/\sigma$. Because radiation of the modes $n < n_{th}$ is suppressed, the coherent synchrotron radiation is possible only for very short bunches:

$$\sigma < \frac{h}{\pi}\sqrt{\frac{3}{2}}\sqrt{\frac{h}{\pi R}} . \quad (7)$$

## II. Impedance Due to Synchrotron Radiation

The electromagnetic field of the synchrotron radiation of a particle may change the energy of other particles in a bunch. The energy variation is described usually in terms of the wake field or, in the case of two point-like particles separated by the distance $s$, in terms of the $\delta$- functional wake field $W(s)$. The last is related to the variation of the energy of the second particle $\Delta E$ due to interaction with the tangential to the trajectory component of the field of the first particle. For a periodic motion in a plane $z = 0$ on a circle with the radius R, $\Delta E(s)$ is the variation of the energy per turn due to the azimuthal harmonic of the electric field:

$$\begin{aligned} W(s) &= -\left(\frac{1}{e^2}\right)\Delta E(s) \\ &= -\frac{c}{e}\int_0^T dt E_\phi\left(R, \phi = \frac{\omega_0 t - s}{R, z = 0, t}\right) . \end{aligned} \quad (8)$$

Here $T = 2\pi R/v = 2\pi/\omega_0$ is the revolution period, and $\omega_0/2\pi$ is the revolution frequency. The field $E_\phi(r,\phi,z,t)$ is periodic in time and azimuth:

$$E_\phi(r,z,\phi,t) = \int \frac{d\omega}{2\pi} e^{-i\omega t}\sum_n e^{in\phi} E_n(r,z,\omega) \quad (9)$$

with harmonics

$$E_n(r,z,\omega) = \omega_0 \sum_\nu E_{n,\nu}\delta(\omega - n\nu) . \quad (10)$$

Eqs. (8) and (10) define the wake in terms of the harmonics $E_{n,\nu}(r,z)$:

$$W(s) = -\frac{c}{e} E_{n,n}(R,0)\, e^{-ins/R} . \quad (11)$$

The longitudinal beam impedance for a periodic motion

$$Z(\omega) = Z_n \omega_0 \delta(\omega - n\omega_0) \quad (12)$$

is the Fourier harmonic of the periodic $W(s) = W(s+2\pi R)$ wake:

$$W(s) = \sum_n \frac{Z_n \omega_0}{2\pi} e^{-in\omega_0 s/v} . \quad (13)$$

Hence

$$Z_n = \frac{2\pi R}{e} E_{n,n}(R,0) . \quad (14)$$

Note that $Z_n$ is the average value of the impedance Eq. (12):

$$< Z(\omega) >= \frac{1}{\omega_0}\int_{\omega_0} d\omega Z(\omega) = Z_n$$

where $n = \omega/\omega_0$ and the interval of the averaging is $\omega_0$.

The azimuthal component of the electric field of a point-like charge $e$ moving with velocity $v$ on a circle with the radius $R$ in the (x,y) plane has harmonics

$$\begin{aligned} E^\phi_{n,\nu}(r,z) &= \frac{e\omega_0}{2\pi}\int d\phi e^{-in\phi}\int_0^T dt\, e^{i\nu\omega_0 t} \\ &\times\left\{\frac{ikv}{c}\cos(\omega_0 t - \phi) - \frac{1}{r}\frac{\partial}{\partial\phi}\right\}\frac{e^{ik\rho}}{\rho} \end{aligned} \quad (15)$$

where $\rho = |\vec{r} - \vec{r}(t)|$, $|\vec{r}(t)| = R$, and the integration over $\phi$ is performed on the interval $2\pi$.

The harmonic $E_{n,n}(R,0)$ defines the impedance. Substituting $\rho = 2R\sin(\alpha)$, where $\alpha = 2(\phi - \omega_0 t)$ we obtain:

$$\mathrm{Re}\,\frac{Z(n)}{n} = \frac{Z_0}{2}\int_0^\pi d\alpha[\beta^2\cos 2\alpha - 1]\,\frac{\sin\{2n\beta\sin\alpha - 2n\alpha\}}{\sin\alpha} . \quad (16)$$

Here $\beta = v/c$, and $Z_0 = 120\pi$ Ohm. The modes of interest are the modes $1 << n << \gamma^3$. For such modes the significant contribution to the integral is given by $n\alpha^3 \leq 1$ so that $\alpha >> 1/\gamma$. Eq. (16) takes the form:

$$\mathrm{Re}\,\frac{Z(n)}{n} = Z_0\int_0^\pi \alpha d\alpha \sin\left(\frac{n\alpha^3}{3}\right) = 0.813\frac{Z_0}{n^{2/3}} . \quad (17)$$

Here we use the following value of the integral:

$$\int_0^\pi dx x^{-1/3}\sin x = 1.172 .$$

Hence, for very large $n$ the impedance rolls off as $n^{-2/3}$. For $n < n_{th}$ impedance is exponentially small. Therefore, the impedance has maximum value at $n = n_{th}$. Eq. (17) and Eq. (5) give:

$$\left[\mathrm{Re}\,\frac{Z(n)}{n}\right]_{max} = 223.3\,\frac{g}{R}\ \mathrm{Ohm} \quad (18)$$

which is reasonable close to the result obtained by exact solution [3].

Let us estimate the effect of the finite transverse size $\sigma_\perp$ of a bunch. The estimate obtained above is valid if

$$\sigma_\perp^2 << 4R^2 \sin^2(\alpha) \simeq R^2\ n^{-2/3}\ .$$

For $n \simeq n_{th}$ that gives $\sigma_\perp^2 << Rh$. Otherwise, the estimate has to be modified taking into account the finite size $\sigma_\perp$.

## III. CONCLUSION

The simple approach described the main features of the synchrotron radiation in a vacuum chamber. That opens the possibility of consideration of the effect for more complicated geometries and, hopefully, clarifies the physics of the problem.

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## IV. APPENDIX

The estimate for the maximum value of the impedance

$$\left[\mathrm{Re}\ \frac{Z(n)}{n}\right]_{max} \simeq 300\ \frac{g}{R}\ \mathrm{Ohm}$$

can be obtained from the exact solution obtained in the paper by Warnock and Morton [2] (see their Eq. (2.47)) for a particle moving in a cylindrical pillbox cavity:

$$\frac{Z(n)}{n} = i\pi^2 Z_o \frac{R}{h}$$

$$\times \sum_{\substack{p>1\\(odd)}} \left[\left(\frac{\alpha_p}{\gamma_p}\right)^2 \frac{J_n(\gamma_p R)}{J_n(\gamma_p b)}\ p_n(\gamma_p b, \gamma_p R) \right. \qquad (A1)$$

$$\left. + \left(\frac{\omega R}{nc}\right) \frac{J_n'(\gamma_p R)}{J_n'(\gamma_p b)} s_n(\gamma_p b, \gamma_p R)\right].$$

Here $h = 2g$,

$$\alpha_p = \frac{\pi p}{2g}, \quad \gamma_p^2 = (\frac{\omega}{c})^2 - \alpha_p^2$$

and

$$p_n(x,y) = J_n(x)Y_n(y) - Y_n(x)J_n(y)\ ,$$
$$s_n(x,y) = J_n'(x)Y_n'(y) - Y_n'(x)J_n'(y)\ .$$

The real part of the impedance is given by the zeros of the denominators. Two terms in Eq. (A1) give equal contribution. Thus, we may consider the first term and double the answer. Expanding denominators near the resonance frequencies

$$\frac{\omega}{c} = \sqrt{\left(\frac{\nu_n}{b}\right)^2 + \left(\frac{\pi p}{h}\right)^2}$$

we obtain:

$$\mathrm{Re}\ \frac{Z(n)}{n} = 2\pi^3 Z_o \frac{R}{h}$$

$$\times \sum_{p,\nu_n} \left(\frac{\pi p c}{\nu_n h}\right)^2 \frac{\nu_n}{\omega} \frac{J_n^2(\frac{R}{b}\nu_n)\ Y_n(\nu_n)}{J_n'(\nu_n)}\ \delta(\omega - \omega_{\nu,p})\ . \qquad (A2)$$

Here $\nu_n$ are the roots of the Bessel function $J_n(\nu_n) = 0$. The distance between resonances $\Delta\omega/c \simeq (1/b), (\pi/h)$ is much smaller than the threshold frequency $\omega_{th}/c \simeq (1/h)\sqrt{(R/h)}$. Therefore, the summation may be replaced by the integration. Introducing the polar coordinates $(\omega, \phi)$

$$\frac{\pi p}{h} = \frac{\omega}{c}\cos\phi\ , \qquad \frac{\nu_n}{b} = \frac{\omega}{c}\sin\phi$$

we have

$$\mathrm{Re}\ \frac{Z(n)}{n} = 2\pi^2\ Z_o\ R\ \frac{\omega}{c}$$

$$\times \int_0^\pi d\phi\ \frac{\cos^2\phi}{\sin\phi}\ \frac{J_n^2(n\sin\phi)\ Y_n(n\frac{b}{R}\sin\phi)}{J_n'(n\frac{b}{R}\sin\phi)}\ .$$

For $n >> 1$

$$J_n(n\sin\phi) \simeq \frac{\cos\phi}{\pi\sqrt{3}}\ K_{1/3}\ \left(\frac{n}{3}\cos^3\phi\right)\ .$$

The main contribution is given by small $\alpha \equiv \pi/2 - \phi$

$$|\ \alpha\ | \leq (1/n)^{1/3}\ .$$

For $b >> R$ the ratio

$$\frac{Y_n(n\frac{b}{R}\sin\phi)}{J_n'(n\frac{b}{R}\sin\phi)} \simeq 1\ .$$

Hence,

$$\mathrm{Re}\ \frac{Z(n)}{n} = \frac{2}{3}\ Z_o\ R\frac{\omega}{c} \int_{-\pi/2}^{\pi/2} d\alpha\ \frac{\sin^4\alpha}{\cos\alpha}\ K_{1/3}^2\ \left(\frac{n}{3}\sin^3\alpha\right)\ .$$

Replacing $\sin\alpha \simeq \alpha$ and using the integral

$$\int_0^\infty dt t^{2/3}\ K_{1/3}^2(t) = \frac{\sqrt{\pi}}{2^{1/3}}$$

and integrating over $\Delta\omega << \omega$ we obtain finally:

$$\mathrm{Re}\ \frac{Z(n)}{n} = \frac{\sqrt{\pi}}{6^{1/3}}\ \frac{Z_o}{n^{2/3}}\ .$$

Hence, the real part of the impedance rolls off as $n^{-2/3}$ for large $n$. The maximum value is reached at the threshold $n_{th}$ given by Eq. (5). That gives

$$\mathrm{Re}\ \frac{Z(n)}{n} = 268\ \frac{g}{R}\ \mathrm{Ohm}\ .$$

## V. REFERENCES

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