Impedance for a Multi-Cell, Multi-Block Structure*

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INTRODUCTION

An expression for the high-frequency limit of the longitudinal coupling impedance for a structure consisting of a large number of cavities (cells) has recently been derived [1]. In that calculation the cells were identical, azimuthally symmetric, and equally spaced. In the present paper we extend the analysis to the case where the cells are grouped into blocks, which themselves are identical and equally spaced, a geometry frequently encountered in multi-cavity structures. The space between blocks is modeled as missing cells. Specifically, we consider the picture in which every $(\alpha-1)$ cells (which form one block) are followed by β missing cells. The total number of cells in the system, both present and missing, is N. The total number of blocks is M. With the definition $\gamma \equiv \alpha - 1 + \beta$, we write $M = N/\gamma$. The separation between the centers of cells is denoted by L, and the separation between the centers of blocks by $\mathcal{L}$, $\mathcal{L} = \gamma L$. From ref. 1, we carry over the assumptions that the cells are well separated from each other, $L \gg g$, where g is the axial extent of one cell, and that $ka^2/L \gg 1$. In the latter condition, k is the wavenumber of radiation, and a the beam pipe radius.

Before embarking on our derivation we show how one can obtain an approximate answer intuitively by a repeated use of the results of ref. 1. If the number of cells in one block is large ($\gamma - \beta \gg 1$), then from ref. 1 we can write the admittance of a single block as

$$[Z_0 Y(k)]_{\mathrm{b}} = (\gamma-\beta)^{-1}\Big(F_0 + K_{\mathrm{c}}\sqrt{\gamma-\beta-1}\tan^{-1}\frac{K_{\mathrm{c}}}{2\sqrt{\gamma-\beta}}\Big). \tag{1}$$

Here F_0 is the admittance of a single cell, K_{c} is a constant pertaining to one cell,

$$F_0 = (1-i)\pi a\sqrt{\pi k/g}\ ; \quad K_{\mathrm{c}} = (1-i)a\sqrt{\pi k/L}\ , \tag{2}$$

and Z_0 is the impedance of free space, $Z_0 = 120\pi\ \Omega$. On the other hand, if the number of blocks is large ($M \gg 1$), then we can write by analogy the admittance of a string of blocks in the same form as Eq. (1), with F_0 replaced by $[Z_0Y(k)]_{\mathrm{b}}$ and K_{c} replaced by K_{b},

$$K_{\mathrm{b}} = (1-i)a\sqrt{\pi k/\mathcal{L}} = K_{\mathrm{c}}/\sqrt{\gamma}\ . \tag{3}$$

We have introduced the assumptions that $\gamma - \beta \ll \gamma$ (the blocks are well separated) and that $ka^2/\mathcal{L} \gg 1$, which are the analogues of $g/L \ll 1$ and $ka^2/L \gg 1$ for single cells. Combining the expressions for a single block and a string of blocks, we get for the total longitudinal impedance of the system

$$\frac{Z(k)}{Z_0} = N(1-\beta/\gamma)\Big[F_0 + (K_{\mathrm{c}}\pi/2)\sqrt{\gamma-\beta} + K_{\mathrm{c}}(1-\beta/\gamma)\sqrt{N}\tan^{-1}\frac{K_{\mathrm{c}}}{2\sqrt{N}}\Big]^{-1}. \tag{4}$$

This is the same as an approximate result which can be obtained through a more formal derivation.

DERIVATION AND ANALYSIS

We begin a more systematic derivation by modifying Eq. (2.11) of ref. 1 to include missing cells. Thus,

$$y_n = \Big(1 - \sum_{m=1}^{n-1} c_{n-m}y_m\Big)\Big(1 - \sum_{k=1}^{\infty}\sum_{s=0}^{\beta-1}\delta_{n,k\gamma-s}\Big)\ , \tag{5}$$

where c_n is

$$c_n = \frac{(1+i)}{a}\sqrt{\frac{g}{\pi k}}\sum_{s=1}^{\infty}\exp\big(\frac{-inLj_s^2}{2ka^2}\big), \tag{6}$$

and j_s are the zeroes of $J_0(x)$. The longitudinal impedance of the system is given by

$$\frac{Z(k)}{Z_0} = \frac{(1+i)}{2\pi a}\sqrt{\frac{g}{\pi k}}\sum_{n=1}^{N} y_n\,. \tag{7}$$

Therefore, our task is to solve Eq. (5) for y_n, or their sum over n, and substitute the result into Eq. (7) to obtain $Z(k)$. The presence of additional terms, the ones containing the delta functions in Eq. (5), however, makes this calculation more difficult than the corresponding one in ref. 1.

To work on Eq. (5), we let $n = j\gamma - r$, where $j = 1, 2, \ldots\infty$, and $r = 0, 1, 2, \ldots\gamma-1$. Thus the index j labels blocks, whereas r labels cells within one block. We also introduce a sequence of generating functions $w_r(h)$ and their sum $w(h)$ given by

$$w_r(h) = \sum_{j=1}^{\infty} h^{j\gamma-r}y_{j\gamma-r}\ ; \quad w(h) = \sum_{r=0}^{\gamma-1} w_r(h). \tag{8}$$

Now our task can be reformulated as follows: find $w(h)$ (rather than all y_n's). A knowledge of this quantity will

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enable us to obtain, under suitable approximations, an expression for the sum appearing in Eq. (7). Multiplying Eq. (5) by $h^{j\gamma-r}$, summing over j, and using $\delta_{n,k\gamma-s} = \delta_{jk}\delta_{rs}$ and $\sum_{m=1}^{j\gamma-r-1} = \sum_{j'=1}^{j-1}\sum_{r'=0}^{\gamma-1} + \delta_{jj'}\sum_{r'=r+1}^{\gamma-1}$, where $m = j'\gamma - r'$, we obtain

$$w_r + (1-\sum_{s=0}^{\beta-1}\delta_{rs})\Big[\sum_{r'=0}^{\gamma-1} w_{r'}(\sum_{j=1}^{\infty} h^{j\gamma-(r-r')}\times c_{j\gamma-(r-r')}) + \sum_{r'=r+1}^{\gamma-1} w_{r'}h^{-(r-r')}c_{(r'-r)}\Big] = (1-\sum_{s=0}^{\beta-1}\delta_{rs})\frac{h^{\gamma-r}}{1-h^{\gamma}}. \quad (9)$$

These are γ coupled linear algebraic equations for the quantities w_r. As is immediately obvious, the first β of these are 0, i.e. $w_0 = w_1 = \ldots = w_{\beta-1} = 0$. We are thus left with $\alpha - 1$ unknown quantities $w_\beta, \ldots, w_{\gamma-1}$, and the same number of equations which are not identically zero.

Evidently, the number of equations for w_r is the same as the number of cells in a block. In the case where this number is small the best way to obtain $w(h)$ is to solve Eqs. (9) directly, thus avoiding any further approximations. In many applications, however, it may not be practical to do this. Hence we now turn to the derivation of an explicit expression for $w(h)$.

We sum Eqs. (9) over r to get

$$w + \sum_{r=\beta}^{\gamma-1}\sum_{r'=\beta}^{\gamma-1} w_{r'}A_{r-r'} + \sum_{r=\beta}^{\gamma-1}\sum_{r'=r+1}^{\gamma-1} w_{r'}B_{r-r'} = \frac{h(1-h^{\gamma-\beta})}{(1-h)(1-h^{\gamma})}. \quad (10)$$

Here

$$A_r = \frac{(1+i)}{a}\sqrt{\frac{g}{\pi k}}h^{\gamma-r}\times \sum_{s=1}^{\infty}[1-h^{\gamma}\exp(\frac{-i\gamma Lj_s^2}{2ka^2})]^{-1}\exp[\frac{-i(\gamma-r)Lj_s^2}{2ka^2}], \quad (11)$$

$$B_r = \frac{(1+i)}{a}\sqrt{\frac{g}{\pi k}}h^{-r}\sum_{s=1}^{\infty}\exp(\frac{irLj_s^2}{2ka^2}). \quad (12)$$

In Eqs. (11, 12) the explicit expression for c_n given by Eq. (6) was used. Since we are going to set $h \simeq 1-1/N$, the assumptions $\gamma/N = 1/M \ll 1$, $ka^2/L \gg \gamma$, and $\gamma - \beta \ll \gamma$ lead to a simplification of Eq. (11). Then

$$A_r \simeq \frac{(1+i)}{a}\sqrt{\frac{g}{\pi k}}\sum_{s=1}^{\infty}\Big[1-h^{\gamma}+\frac{i\gamma Lj_s^2}{2ka^2}\Big]^{-1}, \quad (13)$$

which is independent of r. We note that the assumption $\gamma - \beta \ll \gamma$ has allowed us to neglect the next higher order correction in A_r, even though for small values of β that correction is comparable to B_r. Now the second term on the left side of Eq. (10) can be written only in terms of w.

To compute B_r we convert the sum over s to an integral, which is valid since $ka^2/L \gg \gamma$, and set the lower limit to 0. This choice for the lower limit is immaterial, as another choice of order unity produces only corrections which are subdominant to the result given below. Thus,

$$B_r \simeq \frac{i}{\pi}\sqrt{\frac{g}{L}}\frac{1}{\sqrt{r}}. \quad (14)$$

With the introduction of new indices, $m' = \gamma - r'$, $m = \gamma - r$, the third term on the left side of Eq. (10) now becomes

$$\frac{1}{\pi}\sqrt{\frac{g}{L}}\sum_{m=1}^{\gamma-\beta}\sum_{m'=1}^{m-1} w_{\gamma-m'}\frac{1}{\sqrt{m-m'}}. \quad (15)$$

To proceed further, we require $\gamma - \beta \gg 1$. Eq. (15), with $(1/\pi)\sqrt{g/L}$ factored out, can now be approximated by

$$\sum_{m=1}^{\infty}\sum_{m'=1}^{m-1} e^{-m/(\gamma-\beta)}w_{\gamma-m'}\frac{1}{\sqrt{m-m'}} = \sum_{m''=1}^{\infty} e^{-m''/(\gamma-\beta)}\frac{1}{\sqrt{m''}}\sum_{m'=1}^{\infty} e^{-m'/(\gamma-\beta)}w_{\gamma-m'} \simeq \sum_{m''=1}^{\gamma-\beta}\frac{1}{\sqrt{m''}}\sum_{m'=1}^{\gamma-\beta} w_{\gamma-m'} = w\sum_{m''=1}^{\gamma-\beta}\frac{1}{\sqrt{m''}}. \quad (16)$$

The first equality sign follows by the convolution theorem. The third expression, on the other hand, reverses the step taken in going from Eq. (15) to Eq. (16). It should be noted, though, that using $e^{-m/(\gamma-\beta)}$ to approximate the finite upper limit in the sum over m (going from Eq. (15) to Eq. (16)) can introduce a finite error even for arbitrarily large $\gamma - \beta$. For w_r proportional to $r^{-1/2}$ or to $\delta_{r-\beta}$ or independent of r, which can be inferred from Eq. (17) to be the case when one of the terms in the denominator is much larger than the other ones, we have verified numerically that this error is less than 12%. Finally, for simplicity we replace the sum in the final expression in Eq. (16) by an integral.

Eq. (10) can now be solved for $w(h)$. Taking $h \simeq 1-1/N$, so that $w(h)$ simulates $\sum_{i=1}^{N} y_i$ [1], yields

$$w(1-1/N) = N(1-\beta/\gamma)\big[1+(2/\pi)\sqrt{g/L}\sqrt{\gamma-\beta} + (1-\beta/\gamma)\frac{(1+i)}{a}\sqrt{\frac{g}{\pi k}}\sum_{s=1}^{\infty}(\frac{1}{N}+\frac{iLj_s^2}{2ka^2})^{-1}\big]^{-1}. \quad (17)$$

In order to evaluate the sum over s, such that the result is valid in both limits $\gamma M \gg ka^2/L$ and $ka^2/L \gg \gamma M$ (or $N \gg ka^2/L$ and $ka^2/L \gg N$), we use the method of ref. 1. The result for the impedance is then

$$\frac{Z(k)}{Z_0} = N(1-\beta/\gamma)\big[F_0 + (K\pi/2)\sqrt{\gamma-\beta} + K(1-\beta/\gamma)\sqrt{N-1}\tan^{-1}\frac{K}{2\sqrt{N}}\big]^{-1}. \quad (18)$$

Here both F_0 and K are given by Eq. (2) (with subscript c on K dropped). Physically, the second term in parentheses in the expression above describes the interference between cells within one block, whereas the third one describes the interference between cells in different blocks.

We examine Eq. (18) in two limits which are consistent with the approximations that $M \gg 1, ka^2 \gg \mathcal{L}$, and $1 \ll \gamma - \beta \ll \gamma$. In the limit $N \ll ka^2/L$, corresponding to the case when the diffraction waves from the first and last blocks are strongly coupled, Eq. (18) gives $k^{-1/2}$ behavior for the impedance

$$\frac{Z(k)}{Z_0} = M(\gamma-\beta)F_0(k)^{-1}\times$$
$$\left\{1+(1/2)\sqrt{g/L}\sqrt{\gamma-\beta}\left[1+\sqrt{M(\gamma-\beta)/\gamma}\right]\right\}^{-1}. \quad (19)$$

Now we examine the dependence of the impedance on geometric parameters. First, if the blocks and the cells are not far apart in the sense that $M(\gamma-\beta) \gg \gamma$ and $g(\gamma-\beta)/L > 1$, then Eq. (19) yields

$$Z(k)/Z_0 = 2\sqrt{M\gamma L}[(1-i)\pi a\sqrt{\pi k}]^{-1}. \quad (20)$$

Comparing with the impedance expression for a plain cell structure at $M\gamma L \ll ka^2$, it can be seen that the detailed structure of each block is neglected. Second, if the blocks satisfy $M(\gamma-\beta) \ll \gamma$, while the cells in each block are not well separated $(g(\gamma-\beta)/L \gg 1)$, one gets from Eq. (19)

$$Z(k)/Z_0 = 2M\sqrt{(\gamma-\beta)L}[(1-i)\pi a\sqrt{\pi k}]^{-1}. \quad (21)$$

In this case the cells in each block are coupled as a plain cell structure in the regime $(\gamma-\beta)L \ll ka^2$. However, the contributions to the impedance from different blocks simply add up because they are far apart. Finally, when $g(\gamma-\beta)/L \ll 1$ and $M(\gamma-\beta) < \gamma$, we are led from Eq. (19) to

$$Z(k)/Z_0 = M(\gamma-\beta)\sqrt{g}[(1-i)\pi a\sqrt{\pi k}]^{-1}. \quad (22)$$

Here the impedance is the sum of all single cell impedances, due to the fact that all cells and blocks are well separated.

Similarly, in the limit $N \gg ka^2/L$ (and $\gamma L \ll ka^2$), one gets from Eq. (18)

$$Z(k)/Z_0 = M[F_0(k)(\gamma-\beta)^{-1}\times$$
$$(1-(1/2)\sqrt{g/L}\sqrt{\gamma-\beta-1}) - i\pi ka^2/(\gamma L)]^{-1}, \quad (23)$$

where the first term in the denominator in Eq. (23) is simply $Z_C/Z(k)$ for a single block. When $g(\gamma-\beta) \gg L$, the impedance in Eq. (23) exhibits $k^{-3/2}$ behavior in the region $ka^2 \gg \gamma^2 L/(\gamma-\beta)$, and behaves as described in Eq. (21) for $ka^2 \ll \gamma^2 L/(\gamma-\beta)$ when the coupling between blocks is weak.

NUMERICAL SIMULATIONS

We have performed numerical simulations to compare the result given by Eq. (18) with F_C^{-1} factored out (that is, w) with $S(N) = \sum_{i=1}^{N} y_i$, where the y_i's are obtained directly from Eq. (5). The constants c_n are evaluated numerically. The upper limit in the sum over s in Eq. (6) is chosen such that the maximum argument of the exponential is of the order of 4π. This limit is in agreement with the discussion in ref. 2, which identifies the region where $j_s \sim (\frac{ka^2}{nL})^{(1/2)}$ as contributing most significantly to the sum. Whereas changes of the order of 1 in the upper limit do not affect the result appreciably, Eq. (6) becomes meaningless if s is allowed to take on very large values.

Figure 1 shows the real and imaginary parts of S and w in the regime where $ka^2/L \gg N$. The values of β, γ, and k are $\beta = 335$, $\gamma = 350$, $k = 1.0 \times 10^5$. Figure 2, on the other hand, shows that reasonable agreement between S and w is achieved even when the condition $\gamma - \beta \ll \gamma$ is violated. In this case $\beta = 2$, $\gamma = 10$, $k = 1.0 \times 10^5$. For both figures $a = L = 1$, and $g = 0.1$.

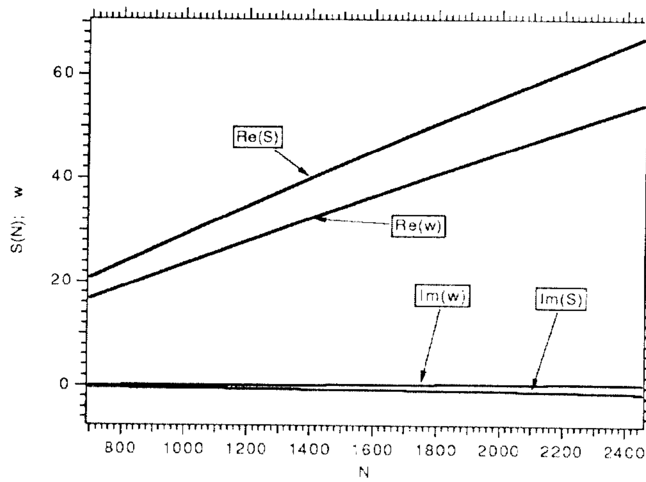

Figure 1: Real and imaginary parts of S and w vs N

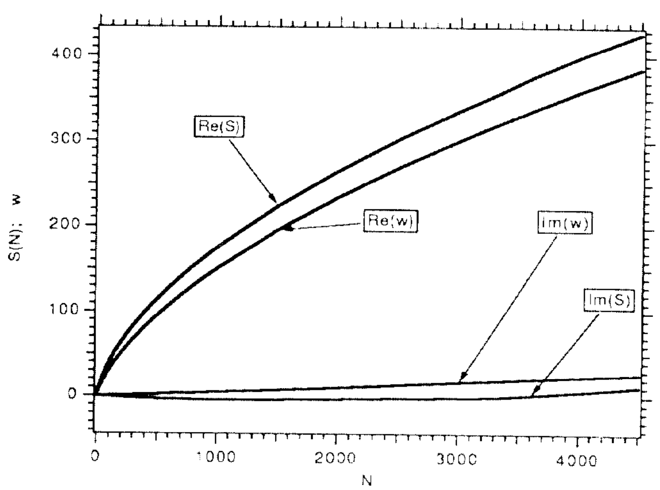

Figure 2: Real and imaginary parts of S and w vs N

REFERENCES

[1] R. L. Gluckstern, IEEE Particle Accelerator Conf. 1157 (1989).
[2] R. L. Gluckstern, Phys. Rev. D **39**, 2773 (1989).