Properties of the Longitudinal Equilibrium Distribution in a Storage Ring*

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Abstract

General properties of the longitudinal equilibrium distribution of a simple model storage ring are discussed using a mapping algorithm for means and correlations of a Gaussian distribution function. The maps for synchrotron oscillations, stochastic excitation, radiation damping and a general localized perturbation are calculated analytically. The fixed point of the concatenated maps is used to characterize the equilibrium distribution.

Introduction

In Ref. [1] a method to investigate the behavior of bunched beams under the influence of localized constant wake forces was introduced. In Ref. [2] this method was extended to resonator type wake forces. In this report we will extend the analysis to the most arbitrary form of a localized interaction.

The single particle map for a localized interaction, in which the longitudinal position does not change, is then given by

$$\begin{aligned} x_1' &= x_1 \ , \\ x_2' &= x_2 + f(x_1, x_2) \ . \end{aligned} \tag{1}$$

Here we introduce the scaled variables $x_1 = \omega_s \tau/\alpha$ and $x_2 = (E - E_0)/E_0$ where ω_s is the synchrotron frequency, α the momentum compaction factor, E_0 the energy of the reference particle, and $c\tau$ the distance between an electron and the reference electron. Clearly the longitudinal coordinate x_1 does not change.

The next element in our model storage ring describes radiation and damping

$$\begin{aligned} x_1'' &= x_1' \ , \\ x_2'' &= \xi x_2' + \sqrt{(1-\xi^2)}\sigma_0 \hat{P} \ , \end{aligned} \tag{2}$$

with $\xi = e^{-T_0/\tau_E}$, where T_0/τ_E is the ratio of the revolution time and the damping time, σ_0 is the natural energy spread, and $\hat{P}$ is a Gaussian white noise defined by $\langle \hat{P} \rangle = 0$, $\langle \hat{P}^2 \rangle = 1$.

The last element describes synchrotron oscillations and is given by

$$\begin{pmatrix} x_1''' \\ x_2''' \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} , \tag{3}$$

where φ is related to the synchrotron tune ν_s by $\varphi = 2\pi\nu_s$.

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The model storage ring is then defined by the succession of mappings

$$x_1, x_2 \xrightarrow{Map\ 1} x_1', x_2' \xrightarrow{Rad.} x_1'', x_2'' \xrightarrow{Osc.} x_1''', x_2''' \ . \tag{4}$$

For this model we will construct the corresponding mappings for the statistical quantities bunch center X_1, average energy X_2, squared bunch length σ_{11}, squared energy spread σ_{22}, and correlation σ_{12} defined by

$$\begin{aligned} X_i &= \langle x_i \rangle \ , \\ \sigma_{ij} &= \langle (x_i - X_i)(x_j - X_j) \rangle \ . \end{aligned} \tag{5}$$

The acute brackets denote averages with respect to the electron distribution function $\psi(x_1, x_2)$. Here we will use a normalized Gaussian

$$\begin{aligned} \psi(x_1, x_2) = & \frac{1}{2\pi\sqrt{\det\sigma}} \\ & \times \exp\Big(-\frac{1}{2}\sum_{i,j=1}^{2} (\sigma^{-1})_{ij}(x_i - X_i)(x_j - X_j) \Big) \ . \end{aligned} \tag{6}$$

In the next section we will first construct the mappings for the statistical quantities X_i and σ_{ij}. In the following section the full one-turn map defined by Eq. (4) is calculated, and its period-1 fixed point which describes the equilibrium configuration is determined. A discussion of the equilibrium concludes this paper.

Mappings for the Statistical Quantities

The maps for the statistical quantities for the radiation and oscillation part are taken from Ref. [1]. We only quote the results here. First, for the radiation part we have

$$\begin{aligned} X_1'' &= X_1' \quad , \quad X_2'' = \xi X_2' \ ; \\ \sigma_{11}'' &= \sigma_{11}' \quad , \quad \sigma_{12}'' = \xi \sigma_{12}' \ ; \\ \sigma_{22}'' &= \xi^2 \sigma_{22}' + (1-\xi^2)\sigma_0^2 \ ; \end{aligned} \tag{7}$$

then for the oscillator part

$$\begin{aligned} \begin{pmatrix} X_1''' \\ X_2''' \end{pmatrix} &= U \begin{pmatrix} X_1'' \\ X_2'' \end{pmatrix} , \\ \begin{pmatrix} \sigma_{11}'''\sigma_{12}''' \\ \sigma_{21}'''\sigma_{22}''' \end{pmatrix} &= U \begin{pmatrix} \sigma_{11}''\sigma_{12}'' \\ \sigma_{21}''\sigma_{22}'' \end{pmatrix} U^T , \end{aligned} \tag{8}$$

where U is the matrix appearing in the single particle map for the oscillation part, Eq. (3). The calculation for the map through the interaction f is rather tedious and

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the details are reported in Ref. [2]. Here we only quote the results

$$
\begin{aligned}
X_1' &= X_1 \ , \\
X_2' &= X_2 + R_1(X,\sigma) \ , \\
\sigma_{11}' &= \sigma_{11} \ , \\
\sigma_{12}' &= \left(1 + R_{22}(X,\sigma)\right)\sigma_{12} + R_{21}(X,\sigma)\sigma_{11} \ , \\
\sigma_{22}' &= \left(1 + 2R_{22}(X,\sigma)\right)\sigma_{22} \\
&\quad + 2R_{21}(X,\sigma)\sigma_{12} + R_3(X,\sigma) \ .
\end{aligned} \tag{9}
$$

The functions R are given in terms of integrals over the interaction function f. They still depend on the quantities (X,σ) before the interaction and are given by

$$
\begin{aligned}
R_1(X,\sigma) &= \frac{1}{\pi}\int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2 \ e^{-t_1^2-t_2^2} \ f(y_1,y_2) \ , \\
R_{21}(X,\sigma) &= \frac{1}{\pi}\int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2 \ e^{-t_1^2-t_2^2} \ \frac{\partial f}{\partial y_1}(y_1,y_2) \ , \\
R_{22}(X,\sigma) &= \frac{1}{\pi}\int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2 \ e^{-t_1^2-t_2^2} \ \frac{\partial f}{\partial y_2}(y_1,y_2) \ , \\
R_3(X,\sigma) &= \frac{1}{\pi}\int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2 \ e^{-t_1^2-t_2^2} \ f^2(y_1,y_2) \\
&\quad - R_1^2(X,\sigma) \ .
\end{aligned} \tag{10}
$$

The argument y_1 and y_2 of the functions $f(y_1,y_2)$ are related to (X,σ) by

$$
\begin{aligned}
y_1 &= X_1 + u_{11}\sqrt{2\lambda_1}t_1 + u_{21}\sqrt{2\lambda_2}t_2 \ , \\
y_2 &= X_2 + u_{12}\sqrt{2\lambda_1}t_1 + u_{22}\sqrt{2\lambda_2}t_2 \ ,
\end{aligned} \tag{11}
$$

where λ_i and $(u_i)_j$ are the eigenvalues and eigenvectors of σ_{ij}. They are calculated in Ref. [2].

$$
\begin{aligned}
\tilde{X} &= \frac{\sigma_{22}-\sigma_{11}}{2\sigma_{12}} \ , \quad \tilde{Y} = \sqrt{1+\tilde{X}^2} \ ; \\
y_1 &= \tilde{X} + \tilde{Y} \quad , \qquad y_2 = \tilde{X} - \tilde{Y} \ ; \\
\lambda_{1,2} &= \frac{\sigma_{11}+\sigma_{22}}{2} \pm \sigma_{12}\tilde{Y} \ ; \\
u_{11} &= \frac{1}{\sqrt{1+y_1^2}} \quad , \quad u_{12} = \frac{y_1}{\sqrt{1+y_1^2}} \ ; \\
u_{21} &= \frac{1}{\sqrt{1+y_2^2}} \quad , \quad u_{22} = \frac{y_2}{\sqrt{1+y_2^2}} \ .
\end{aligned} \tag{12}
$$

The interpretation of the functions R_i in Eq. (9) is straightforward. R_1 is a generalized loss factor, because it describes the energy exchange (it changes the energy-like variable x_2) in the interaction f. R_{22} describes the damping (or anti-damping) effect of the interaction f, because negative (positive) R_{22} decreases (increases) the magnitude of σ_{12} and σ_{22} in Eq. (9). Note from the third line of Eq. (10) that R_{22} is proportional to the derivative of f with respect to the energy-like variable x_2. Therefore damping or anti-damping can only come from an energy dependence of the interaction f. From Eq. (9) we see that R_{21} drives the cross term σ_{12} and is responsible for correlations between energy and position inside the bunch. Note that R_{21} is proportional to the derivative of f with respect to the position-like variable x_1. Therefore an interaction f that distinguishes different parts inside a bunch can produce a correlation between position and energy inside the bunch. Finally, R_3 is a noise term. It can be shown that it is positive definite and increases the energy spread σ_{22}; i.e., it introduces noise.

The concatenated mapping and its fixed point

In order to calculate the one-turn map for the model storage ring defined by Eq. (4) we will follow Ref. [1] and use $\det\sigma$, $\mathrm{Tr}\,\sigma$, and σ_{11} as the mapped quantities. This "trick" facilitates the algebra, because $\det\sigma$ and $\mathrm{Tr}\,\sigma$ are invariant under oscillations and the other maps leave σ_{11} invariant. After a considerable amount of algebra we obtain for the one-turn map

$$
\begin{aligned}
X_1''' &= X_1\cos\varphi + \xi\left(X_2 + R_1(X,\sigma)\right)\sin\varphi \ , \\
X_2''' &= -X_1\sin\varphi + \xi\left(X_2 + R_1(X,\sigma)\right)\cos\varphi \ , \\
\sigma_{11}''' &= \sigma_{11}\cos^2\varphi \\
&\quad + 2\xi[(1+R_{22})\sigma_{12} + R_{21}\sigma_{11}]\sin\varphi\cos\varphi \\
&\quad + [\xi^2(1+2R_{22})\sigma_{22} + 2\xi^2 R_{21}\sigma_{12} + \xi^2 R_3 \\
&\qquad + (1-\xi^2)\sigma_0^2]\sin^2\varphi \ , \\
\det\sigma''' &= \xi^2\left[(1+2R_{22})\det\sigma + R_3\sigma_{11}\right. \\
&\qquad \left. - (R_{21}\sigma_{11} + R_{22}\sigma_{12})^2\right] + \left(1-\xi^2\right)\sigma_0^2\sigma_{11} \ , \\
\mathrm{Tr}\,\sigma''' &= \mathrm{Tr}\,\sigma - \left(1-\xi^2-2\xi^2 R_{22}\right)\left(\sigma_{22}-\sigma_0^2\right) \\
&\quad + 2\xi^2 R_{22}\sigma_0^2 + 2\xi^2 R_{21}\sigma_{12} + \xi^2 R_3 \ .
\end{aligned} \tag{13}
$$

These maps can now be used to investigate the time dependence of the model storage ring as was done in Ref. [2] for a resonator type wake. Here we will investigate the equilibrium configuration. To this end we will calculate the period-1 fixed point of the maps given by Eq. (13). Equating the primed and the triple primed quantities we get the following implicit set of equations for the equilibrium values X_i^∞ and σ_{ij}^∞, and obtain

$$
\begin{aligned}
X_1^\infty &= \frac{\xi}{1+\xi} R_1 \cot\frac{\varphi}{2} \ , \\
X_2^\infty &= -\frac{\xi}{1+\xi} R_1 \ , \\
\sigma_{11}^\infty &= \sigma_{22}^\infty - 2\sigma_{12}^\infty \cot\varphi \ , \\
\sigma_{12}^\infty &= -\frac{\xi R_{21}}{1+\xi(1+R_{22})}\sigma_{11}^\infty \ , \\
\sigma_{22}^\infty &= \sigma_0^2 + \xi^2 \frac{R_3 + 2R_{22}\sigma_0^2 + 2R_{21}\sigma_{12}^\infty}{1-\xi^2(1+2R_{22})} \ .
\end{aligned}
\tag{14}
$$

In the following section we will discuss some of the interesting features of Eq. (14).

Discussion

The equilibrium energy X_2^∞ in the presence of the interaction is changed proportional to the generalized loss factor R_1 and the position of the bunch center X_1^∞ is shifted accordingly. Note that X_1^∞ and X_2^∞ implicitly depend on the equilibrium values X_i^∞ and σ_{ij}^∞ through $R_1 = R_1(X_i^\infty, \sigma_{ij}^\infty)$. The dependence of the loss factor R_1 on the bunch sizes is therefore taken into account in a self-consistent way.

Furthermore note that the dependence of the equilibrium correlation σ_{12}^∞ is proportional to R_{21}. Eq. (10) shows that R_{21} is proportional to the derivative of the interaction f to the position-like variable x_1. Consequently, for an interaction f that affects all particles inside the bunch in the same way cannot introduce a correlation. Moreover, if σ_{12}^∞ is zero we obtain from the third of Eqs. 14 that the bunch length σ_{11}^∞ is proportional to the energy spread σ_{22}^∞. Another way to state this is: if the interaction f only depends on the energy x_2, the correlation σ_{12}^∞ vanishes and the bunch length is proportional to the energy spread.

An example for an interaction that treats all particles in a bunch in the same way is an amplifier Free Electron Laser (FEL), where a continuous external laser is passed over the transversely undulating electrons. Therefore the amplifier FEL does not produce a correlation between energy and position in the bunch.

On the other hand, the light in an oscillator FEL acquires a pulsed structure due to a mode-locking mechanism produced by the FEL process itself. The light pulses are typically much shorter than the bunch length and therefore affect only part of the electrons. Consequently a correlation between energy and position is generated.

A further example is the wake interaction in which the leading particles in a bunch affect the trailing. Obviously, this will then introduce a correlation σ_{12}^∞ in the bunch.

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References

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