LIMITING DENSITY DISTRIBUTION FOR CHARGED PARTICLE BEAMS IN FREE SPACE

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Abstract

An analytic solution is given to the Vlasov equation describing the time evolution of an axially symmetric beam of charged particles spreading radially under the influence of self fields. The marginal densities as a function of radius and of radial velocity are found to be flat out to the expanding edges of the beam. This solution is shown to have the maximum entropy for a given phase space area. Thus an arbitrary initial distribution will evolve toward this solution as a limiting distribution.

I. INTRODUCTION

A beam of charged particles will expand radially as it propagates in free space due to pairwise-repulsive, internal electric fields. It will be shown that the time-dependent distribution, f(r,v,t), that describes the particle density as a function of radius, r, and radial velocity, v, of an axially symmetric beam approaches a limiting analytic form

$$f_s(r,v,t) =$$

$$A \exp[-(2R^2/3)(v-\alpha r)^2/2\varepsilon^2] \qquad (1)$$

$$\text{with } A = [(12\pi)^{1/2}\varepsilon]^{-1}$$

for r < R where R,α are functions of time (to be defined), ε is the emittance and r,v are phase space coordinates whose maximum extent at time t are R(t) and V(t).

The time-dependent limits R, V (α = V/R) are calculated by solving the envelope equation-of-motion of a test particle on the edge of the beam

$$\frac{d^2R}{dt^2} = \left[\frac{2Ne^2}{4\pi\varepsilon_o m\gamma^3}\right]\frac{1}{R} = \frac{K}{R} \qquad (2)$$

where N is the number of particles per unit length of the beam, e and m are the charge and mass of an individual particle, ϵ_o is the permitivity of free space (rationalized units), and γ is the relativistic factor related to the longitudinal velocity, v_1, of the particles, $\gamma=[1-(v_1/c)^2]^{-1/2}$. The collection of constants inside the bracket of eq.(2) will be designated K, the beam perveance, and set equal to unity (K=1) for convenience in the rest of the discussion.

Solutions [1] to eq. (2) are

$$V^2 = 2\ln\frac{R}{R_o}\ , \quad t = \int_{R_o}^{R}\frac{dx}{V(x)} \qquad (3)$$

R(t) and V(t) are plotted in figure 1 for K = 1. This defines the parameters R, R_o=R(0), V, and α.

II. PROOF

The proof that eq. (1) is the limiting density function is as follows. Define the quantity

$$H(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\ \ln f\ dv\ dr = \langle \ln f\rangle \qquad (4)$$

H is related to the entropy, S, of the system by $H=-S/k_b\,\Omega$ where k_b is Boltzmann's constant and Ω is the

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volume of the system. The Boltzmann H theorem [2] states that H decreases with time

$$\frac{dH}{dt} \le 0 \tag{5}$$

until some asymptotic limit is attained.

It can be shown from probability theory [3] that the form of the distribution function, f, that maximizes the entropy of the system can be constructed from the constraints imposed on the system in the following manner. If each of n system constraints is expressed as an expectation value of a given function equal to a constant

$$\langle h_n \rangle = c_n \tag{6}$$

then the density function with the maximum entropy is

$$f_S = \exp\left(\sum_n a_n h_n\right) \tag{7}$$

if the constants a_n can be found. In the present case we first want f_S to normalize to unity so that we chose $h_1=1$, $c_1=1$. Second, we want to keep emittance constant by making the local width in the v-direction inversely proportional to $\langle r^2 \rangle^{1/2}$. Thus we chose $h_2=(v-\langle v(r)\rangle)^2$, $c_2=\epsilon^2/2\langle r^2\rangle$ where the local mean velocity is defined as

$$\langle v(r) \rangle = \frac{\int v\, f(r,v)\, dv}{\int f(r,v)\, dv} \tag{8}$$

This reasoning leads to eq.(1) as the maximum entropy distribution (hence the subscript S on f) and to the conclusion that any initial distribution will asymptotically approach this analytic form as it seeks to maximize entropy (following the Boltzmann H-theorem). In general the entropy of a beam is proportional to the logarithm of the emittance in agreement with ref. 4.

III. DISCUSSION

A unique property of f_S is that its marginal densities

$$g_v(r) = \int_{-\infty}^{\infty} f\, dv\ , \quad g_r(v) = \int_{-\infty}^{\infty} f\, dr \tag{9}$$

are uniformly distributed, specifically $g_v=(2R)^{-1}$, $g_r=(2V)^{-1}$ for r<R and v<V, respectively. The Coulomb repulsion homogenizes the spatial and velocity density functions out to the edges of the expansion.

The Vlasov equation [1] describes the evolution of a density distribution driven by space charge fields

$$-\frac{\partial f}{\partial t} = v\frac{\partial f}{\partial r} + a\frac{\partial f}{\partial v} \tag{10}$$

where a is the acceleration of a particle at radius r. In general, a is given by

$$a(r) = \frac{1}{r}\,\frac{\int_0^r r g_v\, dr}{\int_0^\infty r g_v\, dr} \tag{11}$$

Eq.(1) satisfies the Vlasov equation since $a = r/R^2$, $\langle v(r)\rangle = \alpha r$, and $\langle r^2\rangle = R^2/3$ for f_S.

Fig. 2 shows an initial phase space distribution and numerical solutions of eq.(10) at two subsequent time intervals. We see how the dynamics have forced a high degree of correlation between the v and r coordinates of the particles while preserving the phase space normalization and area.

If an initial phase space distribution is prepared in the form of eq.(1) emittance will not grow. Any other distribution will increase its emittance as it approaches the limiting distribution.

At very long times the limiting distribution becomes

$$f_S(r,v,t\to\infty) \to \frac{1}{2R}\delta(v-\alpha r) \tag{12}$$

for r<R. Eq.(12) is the density

describing the envelope solution of eq.(2) for an initially uniform spatial, zero emittance distribution.

In summary it has been shown that axially-symmetric charged-particle beams spreading under the influence of space charge forces evolve toward a limiting phase space density function, eq.(1), which has the maximum entropy for a given emittance. The same analysis can be applied to spherical charge distributions.

IV. REFERENCES

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[2] K. Huang, 'Statistical Mechanics', John Wiley & Sons, New York, 1963.

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[4] J.D. Lawson, P.M. Lapostolle, and R.L. Gluckstern, Particle Accel. **5**, 61 (1973).

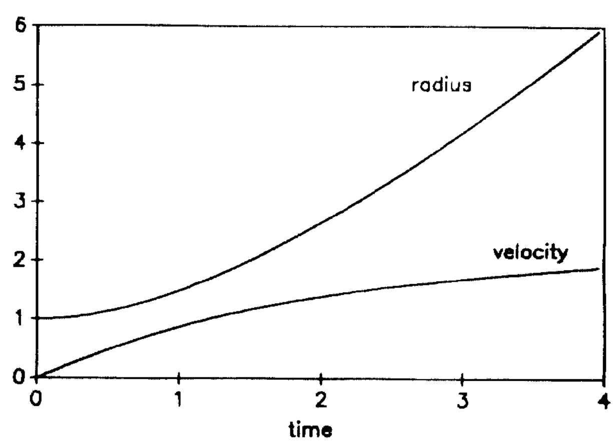

Figure 1. Envelope radius, R, and radial velocity, V, as a function of time. The units are: radius (R_o), velocity ($K^{1/2}$), and time ($R_o/K^{1/2}$).

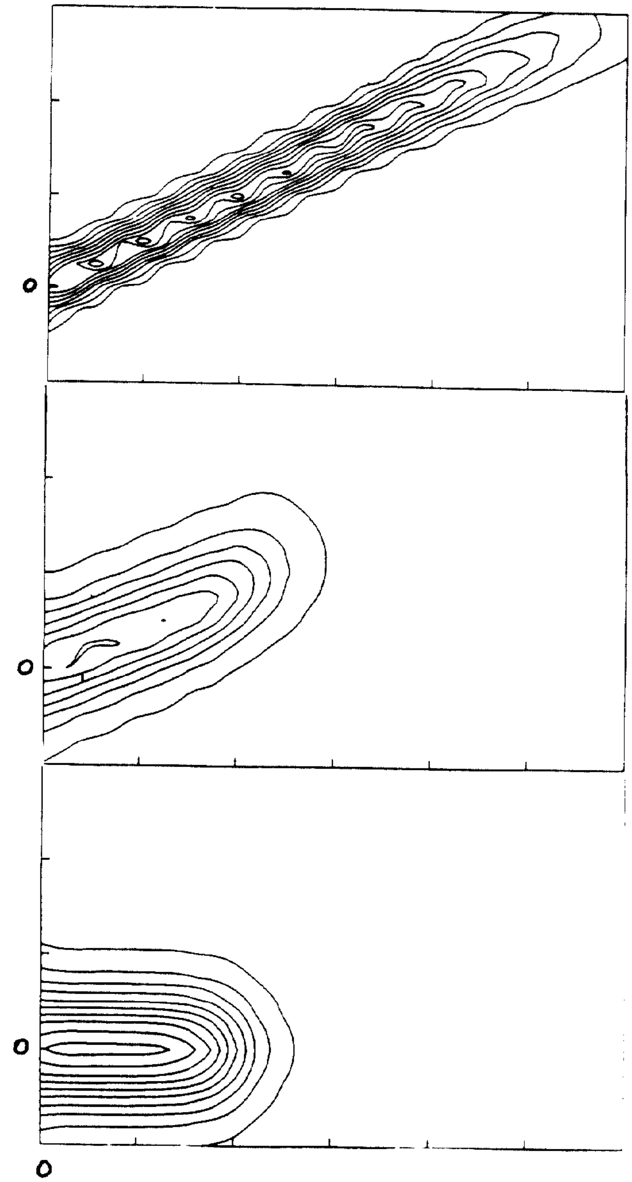

Figure 2. Phase space plots of the beam density f(r,v,t) at three times. Highest density is in the center contour. The horizontal axes are radial extension (only positive values of this cylindrically symmetric distribution are shown); the vertical axes are radial velocity.