Constant of Motion and Dynamic Equations for One Dimensional Autonomous System, and Radiation Damping

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Abstract

The dynamic equations for a one-dimensional autonomous system, along with their dynamic transformations, are expressed in terms of the Constant of Motion of the system. The relation between Hamilton's equations and the Constant of Motion is studied. This approach is applied to the radiation damping suffered by a proton inside the beam circulating around the Superconducting Super Collider (SSC) ring.

I. Introduction

It is well known that any motion of a single particle moving in one-dimensional space can be described by Newton's equation

$$d^2x/dt^2 = F \quad , \tag{1}$$

where the mass of the particle has been included in the definition of the external force, F, "x" is the position of the particle, and "t" represents the time. If F does not depend explicitly on time, we say that the system is "autonomous," and Equation (1) can be written as the following "Autonomous Dynamical System" (ADS):

$$dv/dt = F(x,v) \tag{2a}$$

and

$$dx/dt = v \quad , \tag{2b}$$

where v is the velocity of the particle.

The first integral of motion of this system represents the first constant of motion, which is associated with the total energy of the particle. It is in close relation with the Hamiltonian and Lagrangian of the system [1] which, in turn, allows us to represent the motion of the particle in the phase space. This space is the natural space in accelerator physics, and it is used to see what happens with a beam of charged particles (or a single charged particle) traveling in the accelerator. Many dissipated systems (in particular synchrotron radiation damping) can be expressed in the form (2); these systems represent a challenge for a consistent Hamiltonian and Lagrangian formulation and have interest in Classical Mechanic, Electrical Network Theory, Statistic Mechanics, and Quantum Mechanics as well. In order to understand the theoretical problem presented by these types of systems, a new approach will be formulated based on the Constant of Motion for the ADS (2). In this paper, a constant of motion is associated to (2) and the dynamics equations and Hamilton-like's equations are given in terms of it. The changes in these expressions because of a "dynamical transformation" are also studied in conjunction with the requirements to have an action-angle-like transformation. Finally, this approach is used to study the classical radiation damping suffered by a charged particle that is in a circulating beam of a collider ring accelerator.

II. Constant Of Motion

A constant of motion is a function, K, defined in the space $\{x,v\}$ which satisfies the following expression

$$dK/dt = 0 \quad . \tag{3}$$

According to (2), this means that the following Partial Differential Equation of First Order (PDFO) is satisfied:

$$v\ (\partial K/\partial x) + F(x,v)\ (\partial K/\partial v) = 0 \quad . \tag{4}$$

This equation can be solved by the characteristics method [2], where the equations for the characteristic curves are given by

$$dx/v = dv/F(x,v) = dK/0 \quad . \tag{5}$$

The last term in (5) means that the function K is an arbitrary function of the characteristic curve obtained from the solution of the two first terms of (5). This functionality may be selected in such a way that the constant K represents the usual expression for the energy, if the function *F(x,v)* reduces to a function F(x) when one parameter reaches a specific value [1]. This last statement will be clear when the radiation damping example is seen.

III. Dynamic Equations Using K

Let ρ be an arbitrary function defined in the space $\{x,v\}$, and assume K is the constant of motion of the system (2). The time variation of ρ is given by

$$d\rho/dt = v\ (\partial\rho/\partial x) + F(x,v)\ (\partial\rho/\partial v) \quad , \tag{6}$$

but from Equation (4), the function F can be expressed in terms of the partial derivatives of the constant of motion K as

$$F(x,v) = -\Big(\partial K/\partial x\Big)\Big/\Big(\partial K/\partial v\Big) \quad , \tag{7}$$

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where we must have $(\partial K/\partial v) \neq 0$; otherwise, the constant of motion cannot define a relation between x and v. Using (7) in (6), the variation of ρ with respect to time is expressed by

$$d\rho/dt = v\ \{\rho, K\}_{x,v} \Big/ \Big(\partial K/\partial v\Big) \ , \tag{8}$$

where $\{\rho, K\}_{x,v}$ is the following Poisson-like bracket:

$$\{\rho, K\}_{x,v} = (\partial\rho/\partial x)(\partial K/\partial v) - (\partial\rho/\partial v)(\partial K/\partial x) \ . \tag{9}$$

Taking ρ as the functions x and v, respectively, in (8), the dynamic equations in terms of the constant of motion are

$$dx/dt = v \tag{10a}$$

and

$$dv/dt = -v\ \Big(\partial K/\partial x\Big) \Big/ \Big(\partial K/\partial v\Big) \ . \tag{10b}$$

The Hamilton-like's equations can be obtained from (8) and (10) if the condition $v/\Big(\partial K/\partial v\Big) = 1$ is chosen, but this means that the constant of motion must be of the form

$$K = v^2/2 + V(x) \ , \tag{11}$$

where $V(x)$ is an arbitrary function. This expression is the usual total energy of the system when the force does not depend on the velocity. Consequently, for this particular case, the concept of constant of motion, total energy, and Hamiltonian can be used indistinguishably. Equations (4) and (8) are transformed under a change of variable, $x = x(Q,P)$ and $v = v(Q,P)$, as

$$\frac{d\tilde{\rho}}{dt} = \frac{vJ_{Q,P}}{\frac{\partial Q}{\partial v}\frac{\partial \tilde{K}}{\partial Q} + \frac{\partial P}{\partial v}\frac{\partial \tilde{K}}{\partial P}} \left\{\tilde{\rho}, \tilde{K}\right\}_{Q,P} \tag{12}$$

and

$$\left[v\frac{\partial Q}{\partial x} + f\frac{\partial Q}{\partial v}\right]\frac{\partial \tilde{K}}{\partial Q} + \left[v\frac{\partial P}{\partial x} + f\frac{\partial P}{\partial v}\right]\frac{\partial \tilde{K}}{\partial P} = 0 \ , \tag{13}$$

where $J_{Q,P}$ represents the Jacobian of the transformation, the functions $\tilde{\rho}$ and $\tilde{K}$ have been defined as $\tilde{\rho}(Q,P) = \rho(x(Q,P), v(Q,P))$ and $\tilde{K}(Q,P) = K(x(Q,P), v(Q,P))$.

The expression (12) expresses the fact that Constant of Motion is invariant under dynamical transformations. We must notice that the constant of motion, $\tilde{K}$, can also be obtained from the equation $d\tilde{K}/dt = 0$. If the constant of motion is of the form (11), Hamilton-like's equations are obtained to describe the dynamics of the system, and from the relation (12), it is clear that asking for $J_{Q,P} = 1$ is enough, in order to have invariance of the Hamilton-like's equations under dynamical transformations. One transformation of particular interest to the physicist is the action-angle transformation. This one is characterized by the following relations

$$dQ/dt = \omega = constant \tag{14a}$$

and

$$dP/dt = 0 \tag{14b}$$

Using these relations in (12) and using some algebra, the following condition is obtained:

$$P = \Big(v^2/2 + V(x)\Big)/\omega \ . \tag{15}$$

This result means that the action-angle-like transformations are allowed only in the Hamilton-like systems. Furthermore, if the constant of motion of the autonomous system (2) is not of the form (11), an action-angle-like transformation is not a consistent approach. In the following section, the approach presented above will be illustrated by the well-known phenomenon called radiation damping.

IV. Radiation Damping

When a beam of charged particles (electrons or protons) is moving around its orbit in a stored ring accelerator, it suffers many electromagnetic perturbations produced by external sources (magnets, rf-cavities). These perturbations shape, transport, and accelerate the beam along its orbit, and induce oscillation in its motion (betatron, synchrotron oscillations). The charged particles emit electromagnetic radiation any time they are accelerated (tangentially or orthogonally to its motion). This emitted radiation reduces the energy of the beam and that of the individual charged particles in it. The energy lost by the beam can be compensated using a little bit more rf-power to maintain the beam in its designed orbit. The motion of the individual charged particles of the beam is damped relative to the synchronous particle in the beam [3], bringing about a reduction on the beam phase space size (emittance). This is called radiation damping.

The damping effect in the charged particles of the beam can be studied at first approximation as a simple harmonic oscillation in their relative energy, $\mathbf{x} = \mathbf{E} - \mathbf{E_S}$, with $\mathbf{E}$ the energy of the charged and $\mathbf{E_S}$ the energy of the synchronous charge, damped by a friction term proportional to the rate of change in the relative energy per turn, $\mathbf{v} = \mathbf{dx/dt}$. That is, this phenomenon can be described by

$$d^2x/dt^2 + 2\alpha\ (dx/dt) + \Omega_o^2 x = 0 \ , \tag{16}$$

where Ω_o is the circular synchroton frequency that is given in terms of the revolution time, T_o; energy of the synchronous particle, E_s; change of the energy of the particle with respect the synchronous particle, edV/dT; and a constant, α_o (which depends on the magnetic guide field), as $\Omega_o^2 = (dV/dT)e\alpha_o/T_oE_s$; and α is the damping rate coefficient that can be given in terms of the variation of energy lost per turn with respect to the energy of the particle, dU/dE, as $\alpha = (dU/dE)/2T_o$. Equation (16) can be written as the following one-dimensional-autonomous dynamical system:

$$dv/dt = -(\Omega_o^2\ x + 2\alpha v) \tag{17a}$$

and

$$dx/dt = v \quad . \tag{17b}$$

The constant of motion associated with this system is

$$K(x,v) = \frac{1}{2}\left(v^2 + \Omega_o^2 x^2 + 2\alpha x v\right) \exp\left[2\alpha G(\alpha, \Omega_o, \frac{x}{v})\right] \quad . \tag{18}$$

where G is defined as

$$G(\alpha,\Omega_o,\frac{x}{v}) = \begin{cases} \frac{1}{2\sqrt{\Delta}} Log\left[\frac{(\alpha+\Omega_o^2\frac{x}{v})-\sqrt{\Delta}}{(\alpha+\Omega_o^2\frac{x}{v})+\sqrt{\Delta}}\right], & \text{if } \Omega_o^2 < \alpha^2; \\ -\frac{1}{\alpha+\frac{\Omega_o^2 x}{v}}, & \text{if } \Omega_o^2 = \alpha^2; \\ \frac{1}{\sqrt{-\Delta}} Arctan\left[\frac{\alpha+\Omega_o^2\frac{x}{v}}{\sqrt{-\Delta}}\right], & \text{if } \Omega_o^2 > \alpha^2, \end{cases} \tag{19}$$

and Δ is defined as $\Delta = \alpha^2 - \Omega_o^2$. This constant of motion has three cases: strong dissipation case ($\Omega^2 < \alpha^2$), critical dissipation case ($\Omega^2 = \alpha^2$), and weak dissipation case ($\Omega^2 > \alpha^2$). For very weak radiation levels, $\alpha^2/\Omega_o^2 \ll 1$, and using the relations (18) and (19), the constant of motion can be given at first order of approximation in α as

$$K(x,v) = \frac{1}{2}(v^2 + \Omega_o^2 x^2) +$$

$$2\alpha\left[xv + \frac{1}{2\Omega_o}(v^2 + \Omega_o^2 x^2)\arctan(\frac{\Omega_o x}{v})\right] . \tag{20}$$

Thus, even with a very small dissipation term, the autonomous system (17) does not correspond to a Hamilton-like system, and an action-angle-like procedure would be inconsistent.

The curve, $K = constant$, has a gap per cycle in the phase space when the velocity is zero, $v = 0$. This is originated due to the "arctan" function that appears in the constant of motion. The gap size is a measure of the energy lost per cycle-of-oscillation by the charge; it can be calculated by taking the limit on the constant of motion when the velocity goes to zero from both sides. Observe first that since the curve $K = constant$ must be continuous when $v = 0$, there is a change by π in the argument of the "arctan" function phase. The limits from the right and the left can now be calculated. For the very weak radiation level case and where $x > 0$, these limits produce the following expression:

$$(\delta x) = x\left[\sqrt{1 - \frac{6\pi\alpha/\Omega_o}{1 + 3\pi\alpha/\Omega_o}} - 1\right] . \tag{21}$$

For very weak dissipation limit, $\alpha/\Omega_o \ll 1$, it can be written as $(\delta x) = -x\ (3\pi\alpha/\Omega_o)$, and it is not difficult to see that in the case $x < 0$, the jump is given by $(\delta x) = x\ (3\pi\alpha/\Omega_o)$. The numerical value of α/Ω_o for the pp-SSC accelerator at 20 Tev [4] is on the order of 10^{-8}. This means that it will take on the order of 10^8 cycles of oscillations for a proton beam to shrink about 63% of its initial emittance in the SSC accelerator. This shrinking effect in the beam may not continue longer because of the electrostatic repulsion of the charges and the quantum fluctuations of the radiation [3]. These effects require more elaborate study.

V. Conclusions

In order to understand the Hamiltonian formulation problems presented by dissipative systems, a new approach has been formulated based on the Constant of Motion for an Autonomous System. The dynamics equations are expressed in terms of this constant of motion, and the effect of a dynamical transformation on these equations was studied along with the action-angle transformation. This approach is reduced to the "standard" formulation (Energy, Hamiltonian, Poisson bracket, and canonical transformation) when the constant of motion is of the form $\frac{1}{2}v^2 + V(x)$. When this approach is applied to radiating damping, it brings about a constant of motion which has the right nondissipative limit when the dissipated coefficient goes to zero. Using the simplified weak-dissipation case, the variation in the energy per cycle of synchroton oscillation of the particle in the beam was calculated for the SSC. The basic theoretical answer given here is that the radiation damping phenomenon does not correspond to a Hamilton-like system. Thus, an action-angle transformation is not a consistent procedure for this phenomenon.

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VII. References

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