LIEMAP : A PROGRAM FOR EXTRACTING A ONE–TURN SINGLE EXPONENT LIE GENERATOR MAP

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Abstract

We present a method of computing a one–turn map for a storage ring. Each element of the lattice is represented by a matrix for a linear element followed by a Lie generator for a nonlinear multipole kick. All matrices are moved to the front of the lattice and multiplied to form a single matrix. This re-arrangement also changes each nonlinear Lie operator by a similarity transformation. The Campbell–Baker–Hausdorff (CBH) theorem is used to combine successive nonlinear generators into one generator which is then expressed as a perturbative series in the multipole strengths. In principle, the program can be used to compute the CBH series to any desired order. Routines from a differential algebra library Zlib are used to perform all operations. The advantages of this form of the map are (a) It presents a direct physical connection between the degree of nonlinear behaviour and the strengths of the multipoles, (b) faster extraction time for the map compared to the usual Taylor map of similar accuracy, and (c) the map is guaranteed to be symplectic.

INTRODUCTION

The dynamics of single particles in storage rings can be reproduced to the desired accuracy by Taylor map representations of the lattice, even over long time intervals [1]. These maps have the advantage of being quicker to track (by an order of magnitude) than conventional element by element tracking routines such as Teapot and furthermore the maps can by analysed (using the theory of normal forms) to obtain quantities such as the tune shift with amplitude, the smear etc. without the necessity of tracking [2]. The longest part of the process involves extracting the Taylor map from the lattice description. Typically, it takes 12 CPU hours on the Cray to extract an 11th order map representing the SSC collider. Here we present a method for faster extraction of a map for a large ring, based on the exponential representation of a symplectic map [5]. Depending on the model chosen for the lattice (thin-lens, thick-lens etc.) an exact map representation for the lattice can be obtained. However due to computer limitations, in actual use the map must be truncated in some fashion. We choose as our basis for truncation the order of

the multipole strengths so that in effect we represent the lattice as a perturbation series in the multipole strengths. We estimate that with terms up to the fourth order the map should be sufficiently accurate as to agree with element by element tracking routines. Below we outline our algorithm for extracting the map.

ALGORITHM FOR LIEMAP

We choose as our model the thin lens representation of the lattice. Thus each magnetic element is represented by a kick at the center of the element followed by a drift to the center of the next element. The Hamiltonian generating the motion from one magnet to the next may be written as

$$ H = -(1 + \frac{x}{\rho})[(1 + \delta)^2 - p_x^2 - p_y^2]^{1/2} - \frac{p_\tau}{\beta_0} - \frac{eA_s}{p_0} \quad (1) $$

where our phase space coordinates are $(x, p_x, y, p_y, c\tau, p_\tau)$. The momenta have been scaled by p_0, the design momentum and $p_\tau = -(E - E_0)/p_0 c$. The relative momentum deviation $\delta = (p - p_0)/p_0$ is related to p_τ as

$$ \delta = \sqrt{1 - \frac{2p_\tau}{\beta_0} + p_\tau^2} - 1 $$

The longitudinal vector potential A_s can be written as

$$ -\frac{eA_s}{p_0} = [(\frac{1}{\rho^2} - K_1)x^2 + \frac{K_1}{2}y^2 - \frac{x}{\rho}] + V_{NL} \quad (2) $$

ρ is the bending radius at the magnet, K_1 is the focussing strength and V_{NL} is the nonlinear potential due to higher order multipoles. In terms of the usual skew and normal multipole coefficients a_n, b_n it may be expressed as

$$ V_{NL}(x, y) = \frac{eB_0}{p_0} \text{Re} \sum_{n=2}^{N} \frac{b_n + ia_n}{n}(x + iy)^{n+1} \quad (3) $$

B_0 is the design bending field. In the kick approximation V_{NL} may be regarded as the potential due to an impulsive force at the center of the magnet. For a large ring such as the SSC it is an excellent approximation to expand the terms in the Hamiltonian within the square root in equation (1) and keep only the terms linear in p_x^2 and p_y^2. With this approximation the Hamiltonian may be written as a sum of three terms

$$ H = H_{DR} + H_{LK} + V_{NL} \quad (4) $$

where

$$H_{DR} = \frac{p_x^2 + p_y^2}{2(1 + \delta)} - (\delta + \frac{p_\tau}{\beta_0}) \qquad (5)$$

generates the drift between the magnetic elements and H_{LK} generates the linear kick due to a dipole or quadrupole. For a dipole H_{LK} is given by

$$H_{LK} = \frac{x^2}{2\rho^2} - \frac{\delta x}{\rho} \qquad (6)$$

while for a quadrupole it is given by

$$H_{LK} = -\frac{K_1}{2}x^2 + \frac{K_1}{2}y^2 \qquad (7)$$

Let $\vec{z}_k$ represent the phase space vector at the element k. Then the motion to the next element is determined by the evolution operator $\exp(- : sH :)$, s being the coordinate along the path and $: H := \{H, \}$, the Poisson bracket operator. Hence

$$\vec{z}_{k+1} = \exp(- : H_{DR}(\vec{z}_k)D :)\exp(- :(H_{LK}(\vec{z}_k)+V_{NL}(\vec{z}_k))L :)\vec{z}_k \qquad (8)$$

where D is the drift length from element k to element k+1 and L is the length of the kth magnetic element. Since H_{DR} and H_{LK} are quadratic polynomials in the transverse variables they generate linear transformations in the transverse phase space and consequently may by represented by matrices M_{DR} and M_{LK} respectively when acting on this space. The transformation of the time of flight may be evaluated independently as none of the other variables depend on this variable except at a rf cavity. Since most of the computational effort is spent in updating the transverse variables we will concentrate on the tranformation of these variables in what follows. From the above we see that if $\vec{\eta}_k$ represents the five dimensional phase space vector $(x_k, p_{xk}, y_k, p_{yk}, \delta_k)$ at element k, then

$$\vec{\eta}_{k+1} = \mathcal{M}_{DR}(\vec{\eta}_k)\exp(: -V_{NL}(x_k, y_k))L :)\mathcal{M}_{LK}(\vec{\eta}_k)\vec{\eta}_k \qquad (9)$$

where $\mathcal{M}_{DR}$ and $\mathcal{M}_{LK}$ are the exponential operators generating the drift and linear kick respectively. Combining all the transformations for the N elements in the ring and relabelling the subscripts, we find that the lattice may by represented by a map of the form

$$\mathcal{M} = \mathcal{M}_{N+1}e^{:f_N:} \ldots \mathcal{M}_2 e^{:f_1:} \mathcal{M}_1 \qquad (10)$$

where each f_k is a polynomial in the coordinates (x_k, y_k) of the kth magnetic element. As written, this map cannot be computed since each operator is a function of different variables. Instead we must first transform all of them into functions of the same variables, for instance the phase space coordinates at the beginning of the ring. Using the relation

$$e^{:f_2(\vec{\eta}_2):}\vec{\eta}_2 \equiv e^{:f_2(\vec{\eta}_2):}e^{:f_1(\vec{\eta}_1):}\vec{\eta}_1 = e^{:f_1(\vec{\eta}_1):}e^{:f_2(\vec{\eta}_1):}\vec{\eta}_1 \qquad (11)$$

we find that the order of the operators is inverted and the map for the complete ring is

$$\mathcal{M} = \mathcal{M}_1 e^{:f_1:}\mathcal{M}_2 \ldots e^{:f_N:}\mathcal{M}_{N+1} \qquad (12)$$

where now each operator involves functions of the initial phase space vector $\vec{\eta}_1$.

We now wish to combine all nonlinear operators into a single such operator which will be arranged as a perturbative series in the multipole strengths. We move all the linear operators to the right by a sequence of similarity transformations. We also replace the linear operators $\mathcal{M}_k$ by their matrix representations M_k and use the fact that the exponential operators act in the reverse order to their matrix representatives, i.e

$$\mathcal{M}_1(\vec{\eta}_1)\mathcal{M}_2(\vec{\eta}_1)\vec{\eta}_1 = M_2M_1\vec{\eta}_1 .$$

These operations transform the map to

$$\mathcal{M} = e^{:g_1:} \ldots e^{:g_N:} T \qquad (13)$$

where T is the transfer matrix for the linear lattice,

$$T = M_{N+1} \ldots M_2M_1 \qquad (14)$$

and

$$g_k(\vec{\eta}_1) = \mathcal{M}_1 \ldots \mathcal{M}_k f_k(x_1, y_1) = f_k(M_k \ldots M_1(x_1, y_1)) . \qquad (15)$$

At this point we invoke the CBH theorem [3],[4]to combine the product of exponential operators into a single operator. It enables us to write

$$e^{:\epsilon A:}e^{:\epsilon B:} = e^{:C:} \qquad (16)$$

where, up to fourth order in ϵ, C is

$$C = \epsilon(A + B) + \frac{\epsilon^2}{2}\{A, B\} + \frac{\epsilon^3}{12}\{A - B, \{A, B\}\}$$
$$-\frac{\epsilon^4}{24}\{B, \{A, \{A, B\}\}\} + O(\epsilon^5) \qquad (17)$$

Using this expansion the final form of the map can be written as

$$\mathcal{M} = e^{:G:}T \qquad (18)$$

where G is a polynomial in $\vec{\eta}_1$, arranged in powers of the multipole coefficients (b_n, a_n). In principle we could go to higher order in the multipole strengths at the expense of computational time but our recent success with using 11th order Taylor maps for describing the SSC indicates that the non-linearities are sufficiently weak that including up to fourth order terms will more than suffice.

The transfer matrix T is symplectic. It follows that the map $\mathcal{M}$ is symplectic irrespective of the order of the CBH expansion we use, since an exponential operator generates a symplectic mapping [5]. This is a necessary (but obviously not sufficient) property required of a mapping that purports to represent a lattice accurately.

IMPLEMENTING THE ALGORITHM

In creating the map $\mathcal{M}$, the lattice parameters have to be read in from an input lattice file. We use a thin lens description of the lattice written out on a file named Zfile by the program Teapot [6]. It contains such information as

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the bending angle and the horizontal focussing strength at the dipoles, scaled multipole coefficients which include the length of the element, the drift length to the next element and the misalignment errors. Once this is read in, the map is created step by step as follows. Let $\mathcal{M}^{(k)}$ be the map obtained up to the kth element, i.e

$$\mathcal{M}^{(k)} = e^{:G^{(k)}:}\mathsf{T}^{(k)} \tag{19}$$

then at the (k+1)th step the map is

$$\mathcal{M}^{(k+1)} = e^{:G^{(k)}:}e^{:g_{k+1}:}\mathsf{T}^{(k+1)} \tag{20}$$

where

$$
\begin{aligned}
g_{k+1}(\vec{\eta}_1) &= -\frac{B_0 L}{(B\rho)}\mathrm{Re}\sum_{n=2}^{N}\frac{1}{n}(b_{n,k+1}+ia_{n,k+1})Z_{k+1}^{n+1}\\
Z_{k+1} &= X_{k+1}+iY_{k+1}\\
X_{k+1} &= \sum_{j}(\mathsf{M}_{LK}^{(k+1)}\mathsf{T}^{(k)})_{1j}(\vec{\eta}_1)_j\\
Y_{k+1} &= \sum_{j}(\mathsf{M}_{LK}^{(k+1)}\mathsf{T}^{(k)})_{3j}(\vec{\eta}_1)_j\\
\mathsf{T}^{(k+1)} &= \mathsf{M}_{DR}^{(k+1)}\mathsf{M}_{LK}^{(k+1)}\mathsf{T}^{(k)}
\end{aligned} \tag{21}
$$

and $\mathsf{M}_{LK}^{(k+1)}$, $\mathsf{M}_{DR}^{(k+1)}$ are the linear kick and drift matrix respectively at the (k+1)th element. Before going on to the next element the two exponents $G^{(k)}$ and g_{k+1} are combined using the CBH expansion.

The computationally intensive part of this procedure involves the calculation of the Poisson brackets in the CBH expansion. Due to the avalaibility of differential algebra libraries such as DaBerz [7] and Zlib [8], it is now possible to carry out fast computations with polynomials and vector functions of polynomials. All the functions we calculate are polynomials of variables which are vectors in the space spanned by the components of the initial phase space vector $\vec{\eta}_1$. We have used Zlib for these computations. Another aspect of the implementation worth mentioning is that to speed up the computation of g_{k+1} we use a method involving only multiplication of linear polynomials at every step.

The necessity of limiting the computer memory used requires us to truncate the order of the polynomial for the exponent G. At present we have limited it to 10th order. This also limits the power of the higher order multipole coefficients retained, e.g. the coefficient b_9 retained to 4th power would involve a 34th order polynomial but the truncation to 10th order implies that only the coefficients (b_2, a_2) and (b_3, a_3) are retained to the 4th power. Thus our perturbative series contains multipole coefficients to different powers, the maximum power(≤ 4) being determined by the order of the multipole.

We have implemented the above ideas in our program. Other issues which remain to be addressed are incorporating the misalignment errors and efficient computation of the terms in the CBH series. Currently we are trying out this scheme on a test ring with the final goal of applying it to the SSC and also to the high energy booster (HEB).

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