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A High-Order Moment Simulation Model*

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1. Introduction

We consider a beam transport system in which the particle motions in the two transverse directions (x and y) are coupled. The evolution of the beam envelope in such a transport system, with linear dependence of magnetic field on the transverse spatial coordinates, has been considered by Chernin[1] for a mono-energetic particle beam. Here, we are interested in a beam with a distribution in energy in a magnetic field with nonlinear dependence on the spatial coordinates. Under the first condition, the spatial coordinate (z) along the beam motion cannot be confused with the time variable. Since fast particles will overtake the slower particles as time evolves, the beam envelope is a function of both z and time (t). Therefore, the beam envelope equations are a set of partial differential moment equations, instead of a set of ordinary differential moment equations. It is this aspect of the approach described here that distinguishes itself from the work of Chernin[1] and that of Channell and coworkers[2,3]. The theory by Channel et al. uses moment equations to model a three-dimensional beam bunch. For long bunches, however, it is impractical to carry high enough longitudinal moments to model the oscillations within the bunch. In the following we will derive a relativistically covariant form of moment equations, based on the work of Newcomb[4] and Amendt and Weitzner[5].

2. Relativistic Formuation

We start with a time coordinate t and local Cartesian space coordinate (x^1, x^2, x^3), replacing the usual coordinate (x, y, z), where x^3 is measured along the beam motion direction and x^1 and x^2 are the transverse directions. We define $x^4 = ct$, where c is the speed of light, so that space-time is parametrized by $x^\mu, \mu = 1,2,3,4$. We use a summation convention, and we assume that Latin subscripts and superscripts, i, j, k, l, are summed from one to three, while Greek subscripts and superscripts are summed from one to four. The space-time metric $(ds)^2 = dx^i dx^i - c^2(dt)^2$ becomes $(ds)^2 = dx^\mu dx^\nu g_{\mu\nu}$, where the non-zero elements of the metric tensor $g_{\mu\nu}$ are $g_{ij} = \delta_{ij}$ and $g_{44} = -1$. The metric tensors $g_{\mu\nu}$ and $g^{\mu\nu}$, which is defined so that $g_{\mu\nu} g^{\nu\lambda} = \delta_\mu^\lambda$, may be used to to raise and lower indices covariantly. The usual three velocity v^i may be extended to a relativistic covariant four-velocity u^μ by the definitions $\gamma^{-2} = 1 - v^i v^i/c^2$ and $u^i = \gamma v^i, u^4 = \gamma c$ so that $u^\mu u_\mu = -c^2$.

The electromagnetic field tensor $F_{\mu\nu}$ is antisymmetric and is given in terms of $\vec{E}$ and $\vec{B}$ as

$$E_i = cF_{i4} = -cF_{4i},$$
$$B_1 = F_{23} = -F_{32}, B_2 = F_{31} = -F_{13}, B_3 = F_{12} = -F_{21},$$

while the Lorentz force on a particle of charge q is $q(\vec{E} + \vec{v} \times \vec{B})_i = qF^{i\mu}u_\mu/\gamma$. The general form of the external magnetic field we are interested can be expressed as:

$$\begin{aligned} B_1 &= B_{10} + B_{11}x^1 + B_{12}x^2 \\ &\quad + B_{111}x^1x^1 + B_{112}x^1x^2 + B_{122}x^2x^2, \\ B_2 &= B_{20} + B_{21}x^1 + B_{22}x^2 \\ &\quad + B_{211}x^1x^1 + B_{212}x^1x^2 + B_{222}x^2x^2, \\ B_3 &= B_{30} + B_{31}x^1 + B_{32}x^2 \\ &\quad + B_{311}x^1x^1 + B_{312}x^1x^2 + B_{322}x^2x^2; \end{aligned}$$

where all the coefficients $B_{10}, B_{20}, B_{30}, B_{11}, ..., B_{222}$ are functions of x^3, with B_{i0} the dipole, B_{ij} the quadrupole, and B_{ijk} the sextupole components. The beam distribution function, $f(x^\mu, u^i)$, satisfies the relativistic Vlasov equation:

$$\left[\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla} + \frac{q}{m}\left(\vec{E} + \vec{v} \times \vec{B}\right) \cdot \frac{\partial}{\partial \vec{u}}\right] f = 0 \quad , \tag{1}$$

where m is the particle mass. In a covariant formulation, we can multiply Eq. 1 by γ and rewrite the relativistic Vlasov equation as

$$\left(u^\mu \frac{\partial}{\partial x^\mu} + \frac{q}{m} F^{i\mu} u_\mu \frac{\partial}{\partial u^i}\right) f = 0 \quad . \tag{2}$$

The volume element $d\omega = du^1 du^2 du^3/\gamma$ in the four-momentum space is invariant under a Lorentz transformation. Since the transverse coordinates, x^1 and x^2, are invariant under a Lorentz transformation, we define an invariant phase space volume element under a Lorentz transformation to be $d\Omega = dx^1 dx^2 du^1 du^2 du^3/\gamma$, and a phase space average

$$\langle X \rangle = h^{-1} \int X f d\Omega, \tag{3}$$

with $h = \int f d\Omega$. The lowest moment of the Vlasov equation (Eq. 2) gives

$$\frac{\partial}{\partial x^3} h\langle u^3 \rangle + \frac{\partial}{\partial x^4} h\langle u^4 \rangle = 0. \tag{4}$$

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With Eq. 2 multiplied by u^ν and $u^\nu u^\lambda$ then integrated over $d\Omega$, we have:

$$\frac{\partial}{\partial x^3}h\langle u^3u^\nu\rangle + \frac{\partial}{\partial x^4}h\langle u^4u^\nu\rangle = \frac{q}{m}h\langle F^{\nu\mu}u_\mu\rangle, \quad (5)$$

and

$$\frac{\partial}{\partial x^3}h\langle u^3u^\nu u^\lambda\rangle + \frac{\partial}{\partial x^4}h\langle u^4u^\nu u^\lambda\rangle = \frac{q}{m}h(\langle F^{\nu\mu}u_\mu u^\lambda\rangle + \langle F^{\lambda\mu}u_\mu u^\nu\rangle), \quad (6)$$

respectively. There are four independent equations represented in Eqs. 5 and ten in Eqs. 6. Equations 4 to 6 are basically the same as the fluid equations of Newcomb[4] and Amendt and Weitzner[5], with the additional averaging over the transverse coordinates. If $F^{\mu\nu}$ is independent of the transverse coordinates, then Eqs. 4 to 6 can be reduced to a close system by assuming the third order correlations are negligible, which is the standard approximation used in truncating most fluid equations. Since $F^{\mu\nu}$ depends on the transverse coordinates, Eqs. 5 and 6 cannot be closed without introducing the spatial moment equations:

$$\frac{\partial}{\partial x^3}h\langle u^3x^i\rangle + \frac{\partial}{\partial x^4}h\langle u^4x^i\rangle = h\langle u^i\rangle, \quad (7)$$

$$\frac{\partial}{\partial x^3}h\langle u^3u^\nu x^i\rangle + \frac{\partial}{\partial x^4}h\langle u^4u^\nu x^i\rangle = h\langle u^\nu u^i\rangle + \frac{q}{m}h\langle F^{\nu\mu}u_\mu x^i\rangle, \quad (8)$$

and

$$\frac{\partial}{\partial x^3}h\langle u^3x^ix^j\rangle + \frac{\partial}{\partial x^4}h\langle u^4x^ix^j\rangle = h\langle x^ju^i\rangle + h\langle x^iu^j\rangle, \quad (9)$$

for $i, j = 1, 2$ only.

We also define the following second order correlation functions:

$$[u^\mu u^\nu] = h^{-1}\int f(u^\mu - \langle u^\mu\rangle)(u^\nu - \langle u^\nu\rangle)d\Omega, \quad (10)$$

and similar definitions for the third order correlation functions.

From these definitions, we have:

$$\begin{aligned}\langle u^\mu u^\nu\rangle &= [u^\mu u^\nu] + \langle u^\mu\rangle\langle u^\nu\rangle,\\ \langle u^\mu x^i\rangle &= [u^\mu x^i] + \langle u^\mu\rangle\langle x^i\rangle,\\ \langle x^ix^j\rangle &= [x^ix^j] + \langle x^i\rangle\langle x^j\rangle,\end{aligned}$$

$$\begin{aligned}\langle u^\mu u^\nu u^\lambda\rangle =& [u^\mu u^\nu]\langle u^\lambda\rangle + [u^\mu u^\lambda]\langle u^\nu\rangle + [u^\lambda u^\nu]\langle u^\mu\rangle\\ &+ [u^\mu u^\nu u^\lambda] + \langle u^\mu\rangle\langle u^\nu\rangle\langle u^\lambda\rangle.\end{aligned} \quad (11)$$

and other similar expressions.

With these definitions we can rewrite Eqs. 5 to 9, ignoring all third order correlations, as:

$$\mathcal{D}\langle u^\nu\rangle = \frac{q}{m}\langle F^{\nu\mu}u_\mu\rangle - h^{-1}(\frac{\partial}{\partial x^3}h[u^3u^\nu] + \frac{\partial}{\partial x^4}h[u^4u^\nu]), \quad (12)$$

$$\mathcal{D}\langle x^i\rangle = \langle u^i\rangle - h^{-1}(\frac{\partial}{\partial x^3}h[u^3x^i] + \frac{\partial}{\partial x^4}h[u^4x^i]), \quad (13)$$

$$\begin{aligned}\mathcal{D}[u^\nu u^\lambda] =& \frac{q}{m}(\langle F^{\nu\mu}u_\mu u^\lambda\rangle + \langle F^{\lambda\mu}u_\mu u^\nu\rangle\\ &- \langle u^\lambda\rangle\langle F^{\nu\mu}u_\mu\rangle - \langle u^\nu\rangle\langle F^{\lambda\mu}u_\mu\rangle)\\ &- ([u^3u^\nu]\frac{\partial}{\partial x^3} + [u^4u^\nu]\frac{\partial}{\partial x^4})\langle u^\lambda\rangle\\ &- ([u^3u^\lambda]\frac{\partial}{\partial x^3} + [u^4u^\lambda]\frac{\partial}{\partial x^4})\langle u^\nu\rangle,\end{aligned} \quad (14)$$

$$\begin{aligned}\mathcal{D}[u^\nu x^i] =& \frac{q}{m}(\langle F^{\nu\mu}u_\mu x^i\rangle - \langle x^i\rangle\langle F^{\nu\mu}u_\mu\rangle) + [u^\nu u^i]\\ &- ([u^3u^\nu]\frac{\partial}{\partial x^3} + [u^4u^\nu]\frac{\partial}{\partial x^4})\langle x^i\rangle\\ &- ([u^3x^i]\frac{\partial}{\partial x^3} + [u^4x^i]\frac{\partial}{\partial x^4})\langle u^\nu\rangle,\end{aligned} \quad (15)$$

and

$$\begin{aligned}\mathcal{D}[x^ix^j] =& [x^iu^j] + [x^ju^i] - ([u^3x^i]\frac{\partial}{\partial x^3} + [u^4x^i]\frac{\partial}{\partial x^4})\langle x^j\rangle\\ &- ([u^3x^j]\frac{\partial}{\partial x^3} + [u^4x^j]\frac{\partial}{\partial x^4})\langle u^i\rangle,\end{aligned} \quad (16)$$

where $\mathcal{D} = \langle u^3\rangle\frac{\partial}{\partial x^3} + \langle u^4\rangle\frac{\partial}{\partial x^4}$ is the relativistic convective derivative.

Physical meanings can be attached to the moments appeared in Eqs. 12 to 16. The first order moments $\langle x^1\rangle$ and $\langle x^2\rangle$ denote the centroid position, and $\langle u^1\rangle, \langle u^2\rangle, \langle u^3\rangle$ and $\langle u^4\rangle$ are associated with the beam current and density respectively. The second order spatial correlations $[x^ix^j]$ with $i, j = 1, 2$ define the transverse ellipic beam envelope. The second order momentum correlations $[u^\nu u^\mu]$ are the thermal momentum/energy spread. The second order cross correlations $[x^iu^\nu]$ are the current and density dipole moments.

3. Space Charge Models

A. Chernin's Model

A simple model can be introduced at this point to cast Eqs. 12 to 16 to a concrete form if $F^{\mu\nu}$ is expanded to linear terms of x^1 and x^2. This system of twenty-eight equations is a close set of moment equations for the twenty-eight moments. The desired form of $F^{\mu\nu}$ is obtained by expanding the magnetic field to linear terms of x^1, x^2 and employing Chernin's space charge model[1] for the electric field, which is given by

$$\begin{aligned}E^1 &= \rho(q_{11}x^1 + q_{12}x^2),\\ E^2 &= \rho(q_{12}x^1 + q_{22}x^2),\\ E^3 &= 0,\end{aligned}$$

where ρ is the line charge density of the beam, $q_{11} = S_2/D, q_{22} = S_1/D, q_{12} = -[x^1x^2]/D, D = S_0(S_1 + S_2), S_1 = [x^1x^1] + S_0, S_2 = [x^2x^2] + S_0$, and $S_0 = ([x^1x^1][x^2x^2] - [x^1x^2]^2)^{\frac{1}{2}}$. With this model we can evaluate the moments that involve $F^{\mu\nu}$ in Eqs. 14 to 16. For example, we have the following expressions, including self magnetic field:

$$
\begin{aligned}
\langle F^{1\mu}u_\mu\rangle &= B_{30}\langle u^2\rangle - B_{20}\langle u^3\rangle \\
&\quad - B_{21}\langle x^1u^3\rangle - B_{22}\langle x^2u^3\rangle \\
&\quad + \rho(q_{11}\langle x^1u^4\rangle + q_{12}\langle x^2u^4\rangle)/c/\langle\gamma\rangle^2, \\
\langle F^{2\mu}u_\mu\rangle &= -B_{30}\langle u^1\rangle + B_{20}\langle u^3\rangle \\
&\quad + B_{11}\langle x^1u^3\rangle + B_{12}\langle x^2u^3\rangle \\
&\quad + \rho(q_{12}\langle x^1u^4\rangle + q_{22}\langle x^2u^4\rangle)/c/\langle\gamma\rangle^2, \\
\langle F^{3\mu}u_\mu\rangle &= B_{20}\langle u^1\rangle + B_{21}\langle x^1u^1\rangle + B_{22}\langle x^2u^1\rangle \\
&\quad - B_{10}\langle u^2\rangle - B_{11}\langle x^1u^2\rangle - B_{12}\langle x^2u^3\rangle, \\
\langle F^{4\mu}u_\mu\rangle &= -\rho(q_{11}\langle x^1u^1\rangle + q_{12}\langle x^2u^1\rangle)/c/\langle\gamma\rangle^2 \\
&\quad - \rho(q_{12}\langle x^1u^2\rangle + q_{22}\langle x^2u^2\rangle)/c/\langle\gamma\rangle^2.
\end{aligned}
\tag{17}
$$

Equations 14 to 16 can be related to the second order moment equations of Chernin[1] if we restrict them to the same external magnetic field as in Ref. 1, i.e. $B_{10} = B_{20} = 0$ and no sextupole components. We can identify our notatation $[x^iu^\mu]$ with Σ_{ij} in Ref. 1 by the following rules: $[u^iu^j] = \Sigma_{2i,2j}, [u^ix^j] = \Sigma_{2i,2j-1}, [x^ix^j] = \Sigma_{2i-1,2j-1}$. Since $[u^3u^i] = [u^4u^i] = [u^3x^i] = [u^4x^i] = 0$ (for $i = 1, 2$) for a mono-energetic beam with a delta function distribution in u^3, Eqs. 14 to 16 is the same set of equations as in Ref. 1, with $\mathcal{D}$ equivalent to the ordinary time derivative. Notice that in such a case, the ten second moments (correlations) in the transverse directions form a close system and are no longer coupled to the zeroth, first and other second moments.

B. Cylindrical Model with Image Charges

Chernin's model does not take into account the effects of the image charges of the metallic boundary nor the longitudinal component of the space charge fields. To construct an improved space charge model we assume that the charge density, ρ, is given by a collection of charge units;

$$\rho = \sum_{i=1}^{N} a_i(x_3, t) g(\vec{x} - \vec{x_i}), \tag{18}$$

where g is the distribution of finite size charge elements, e. g. truncated Gaussians, and the location of the charge elements is assumed to be independent of x_3. Note that g depends on x_1, x_2 and x_3, while a_i depends only on x_3. To illustrate how the decomposition is achieved, we use the second order system as an example. To second order we have six spatial moments, therefore we have $N = 6$ and a matrix equation to relate the coefficients a_i with the spatial moments.

$$h \begin{pmatrix} 1 \\ \langle x_1\rangle \\ \langle x_2\rangle \\ \langle x_1x_1\rangle \\ \langle x_1x_2\rangle \\ \langle x_2x_2\rangle \end{pmatrix} = M \begin{pmatrix} a_1 \\ \vdots \\ a_6 \end{pmatrix}, \tag{19}$$

where M is a 6×6 matrix whose elements are of the form

$$\int x_k^m x_l^n g(\vec{x} - \vec{x_i}) dx_1 dx_2,$$

with $k, l = 1, 2$ and $m + n \leq 2$. Equation (19) can be easily inverted to express a_i in terms of the spatial moments.

To calculate the space charge fields due to the charge distribution, we have to find the Green's function inside a metallic cylinder, in the beam frame:

$$G(\vec{x}, \vec{x'}) = \frac{4\pi}{a} \sum_{m,n} \frac{J_m(x_{mn}\frac{r}{a}) J_m(x_{mn}\frac{r'}{a})}{x_{mn}[J'_m(x_{mn})]^2} cos m(\theta - \theta') \cdot exp(-|x_3 - x'_3| \frac{x_{mn}}{a}),$$

where r and θ are the cylindrical counterpart of x_1 and x_2, J_m is the Bessel function, x_{mn} is the nth zero of the Bessel function J_m anf a is the radius of the cylindrical tube. The electrostatic potential in the beam frame becomes

$$\phi(\vec{x}) = \int G(\vec{x}, \vec{x'}) \rho(\vec{x'}) d^3x',$$

assuming the velocity spread in x_3 is not important. Transforming the space charge fields from the beam frame to the laboratory frame we have

$$
\begin{aligned}
F^{1\mu}u_\mu &= E_1/\gamma, \\
F^{2\mu}u_\mu &= E_2/\gamma, \\
F^{3\mu}u_\mu &= E_3\gamma,
\end{aligned}
\tag{20}
$$

where E_i are the components of the space charge electric field in the laboratory frame. After some algebra, the space charge contribution to $\langle F^{\nu\mu}u_\mu\rangle$ can be written as

$$\langle F^{\nu\mu}u_\mu\rangle = h^{-1} \sum_{i,j} a_i(x_3) \int a_j(x'_3) A_\nu(x'_3) d^3x',$$

where $A_\nu(x'_3)$ are time independent precomputed integrals of $g(\vec{x'} - \vec{x_i})$ and the Green's function. In the bend, corrections due to first order toroidal effect can be evaluated using similar technique. It is interesting to note that space charge effects appear as bilinear terms here compared with the more nonlinear behavior in the Chernin model.

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