# RECOVERING PHASE DENSITY DISTRIBUTION FROM LINE DENSITY*

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*Abstract*

We present an algorithm to recover the longitudinal density distribution of the particles in a stationary bunch, from the experimentally obtained line density. This algorithm can be used as an alternative to the analytical theory.

## I. Introduction

The knowledge of particle density distribution in longitudinal phase space is important for the study of various instabilities and for computer simulations, which always start from an assumed initial distribution. This initial density profile should be as close to experiment as possible.

In phase space any particle is characterized by its energy E and phase angle $\varphi$, while density is some function $\rho=\rho(E,\varphi)$ which can also depend on time t. Experimentally we can not directly observe phase density distribution. What we see is the line density $\lambda=\lambda(\varphi)$ which is an integral of $\rho$ over all particles with the given phase angle $\varphi$ :

$$\lambda(\varphi) = \int_{E_{min}(\varphi)}^{E_{max}(\varphi)} \rho(E,\varphi)dE \ . \qquad (1)$$

Thus, the problem arises: *how to recover unknown phase density distribution* $\rho(E,\varphi)$ *from the given line density* $\lambda(\varphi)$ *?*

Generally speaking, this problem has no unique solution. However, there are practical cases where a unique solution $\rho$ can be found. A review of some cases along with analytical theory is found in [1].

In this paper we deal with a stationary bunch where its local density, $\rho$, can be reduced to the function of the Hamiltonian **H**:

$$\rho=\rho(E,\varphi)=\rho(\mathbf{H}) \qquad (2)$$

for which the line density is an even function:

$$\lambda(-\varphi)=\lambda(\varphi). \qquad (3)$$

We will work in the dimensionless phase space $(\delta E,\varphi)$ with normalized energy $\delta E=(E-E_s)/\Delta E_o$ and normalized time $\tau=\Omega_o t$ measured in synchrotron periods; $\Delta E_o$ is the half-height of the bucket. The Hamiltonian of motion below transition energy for the stationary bunch is

$$\mathbf{H}(\delta E,\varphi) = -\delta E^2 - \mathrm{Sin}^2(\varphi/2). \qquad (4)$$

## II. The Building Blocks

The recovering algorithm is composed of a number of *steps*.

Each step consists of several *blocks*: *B1*, *B2*, ... Below we describe each block.

### *B1. The grid*

In phase space $(\varphi,\delta E)$ a bunch of length $2r$ and of height $2\delta E_b$ can be inscribed into a rectangle of size $2r\times 2\delta E_b$. Let $M,N$ be any integer, $L=MN$ and

$$h^r=\frac{r}{M}\ , \quad h^x=\frac{h^r}{N}, \quad h^y=\frac{\delta E_b}{2L}\ . \qquad (5)$$

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Then introducing a grid

$$\varphi_i = h^x \cdot i \ , \quad i=0,\pm1,\pm2,\ldots,\pm L, \qquad (6)$$

$$E_j = h^y \cdot j \ , \quad j=0,\pm1,\pm2,\ldots,\pm 2L, \qquad (7)$$

we will cover the bunch area by small rectangles $\mathcal{R}_{ij}$ whose vertices are $(\varphi_i, E_j)$.

## B2. The rings

We now break the bunch into $M$ elliptic-like rings $\mathcal{E}_k$ bounded by two closed trajectories $\delta E_k(\varphi)$ and and $\delta E_{k-1}(\varphi)$, $k=1,2,\ldots,M$ .

$$\delta E_k = \pm\sqrt{\mathrm{Sin}^2\frac{\varphi_k}{2} - \mathrm{Sin}^2\frac{\varphi}{2}}, \quad -\varphi_k \leqslant \varphi \leqslant \varphi_k \qquad (8)$$

$$\varphi_k = -\tau + h^x \cdot k \ , \quad k=0,1,\ldots,M. \qquad (9)$$

## B3. Random particles

Within any rectangle $\mathcal{R}_{ij}$ we can choose a random (particle) point $\mathcal{P} = \mathcal{P}(\bar{\varphi}_i, \bar{E}_j)$ with coordinates

$$\begin{cases} \varphi_i = \bar{\varphi}_i + h^x \cdot \mathbf{RND}(i), \\ \delta\bar{E}_j = \delta E_j + h^y \cdot \mathbf{RND}(j), \end{cases} \qquad (10)$$

where **RND** is a generator of random numbers homogeneously distributed between 0 and 1:

$$0 \leqslant \mathbf{RND} \leqslant 1 \ . \qquad (11)$$

## B4. The ring covering

For any ring $\mathcal{E}_k$ we can find all rectangles, $\mathcal{R}_{ij}$, intersecting that ring. We denote those rectangles with a bar:

$$\bar{\mathcal{R}}_{ij} = \mathcal{R}_{ij} \cap \mathcal{E}_k \neq \emptyset \ . \qquad (12)$$

Applying to all such rectangles block *B3*, we find a set of random particles. Some of these particles lie within the ring $\mathcal{E}_k$, others lie outside. Those which do lie in $\mathcal{E}_k$ should satisfy:

$$\mathrm{Sin}^2\frac{\varphi_{k-1}}{2} \leqslant \delta\bar{E}_j^2 + \mathrm{Sin}^2\frac{\bar{\varphi}_i}{2} \leqslant \mathrm{Sin}^2\frac{\varphi_k}{2}, \qquad (13)$$

where $\varphi_k$ is determined by (9). Those particles which don't satisfy (13) should be discarded from further treatment. As a result, the ring $\mathcal{E}_k$ will be covered by the set of the particles whose density is almost (approximately) **homogeneous.**

We call *ring cover* this procedure, which leads to establishing a set of particles for the given ring $\mathcal{E}_k$. An arbitrary ring $\mathcal{E}_k$ is composed of small rectangles $\mathcal{R}_{ij}$ many of which are cut by the ring boundaries. Such rectangles are *truncated* contrary to *full* rectangles which are not truncated.

The result of the ring covering is that the every full rectangle contains *one* particle, while some of the truncated rectangles may contain *one* or *none*.

Subject to the covering procedure we will denote a *one-fold* covered ring as $\mathcal{E}_k^1$.

Applying this procedure $q$ times to the same ring we will get a $q$-fold covered ring with the homogeneous particle distribution of higher density. The homogeneity of the distribution is approximate due to the truncated rectangles lying along two borders of the ring.

## B5. The projection

Along with the set of rings, we need some construction connecting those rings with the line density. The latter is usually obtained experimentally as a table. We will assume we have an interpolating algorithm able to evaluate $\lambda(\varphi)$ for any $\varphi$ within $-\tau \leqslant \varphi \leqslant \tau$ , where $\tau$ is the bunch half-length.

As we have seen, altogether we have $M$ rings covering the bunch. Let's consider $k$ consecutive rings ($k \leqslant M$). Some of them can be covered by particles as was described in block *B4*. So we have ring sequence

$$\mathcal{E}_1^{q_1}, \ \mathcal{E}_2^{q_2}, \ \ldots, \ \mathcal{E}_k^{q_k}, \quad q_i \geqslant 0. \qquad (14)$$

Any ring, $i$, has two boundary curves, intersecting the axis $\varphi$ in two pairs of symmetrical points.

On the left-hand side of $\varphi=0$ consider the $k$-th ring intersection with axis $\varphi$, where $\varphi=\varphi_{k-1}$ and $\varphi=\varphi_k$ are both taken from (9). Drawing through these points two vertical

lines, we will intersect the bunch as well as line density graph. Now let's find out how many particles of the bunch lie between the two verticals. These particles come from all $k$ rings. Suppose the total number of these particles is $\mathbf{N}_k$. If the density distribution which was created within the bunch were exactly the same distribution as that from which the experimental $\lambda(\varphi)$ was taken, then we would have $\mathbf{N}_k = \mathbf{L}_k$, where

$$\int_{\varphi_{k-1}}^{\varphi_k} \lambda(\varphi)d\varphi = \mathbf{L}_k \ . \qquad (15)$$

The algorithm in question is aimed to generate a particle distribution which makes the integral in (15) as close to $\mathbf{N}_k$ as possible: $\mathbf{L}_k \cong \mathbf{N}_k$. Our goal is to evaluate $\mathbf{L}_k$ and $\mathbf{N}_k$, for subsequent comparison using other branches of the algorithm. We will assume that along with the interpolating algorithm for $\lambda$ we also have an integrating algorithm to compute any $\mathbf{L}_k$ from the given $\lambda$. Thus all Block 5 requires is to calculate $\mathbf{L}_k, \mathbf{N}_k$ for any given $k$. We call this process *a projection*.

## III. The algorithm

**Step1:** Choose M,N.
**Step2:** *B1*. Define rectangles $\mathcal{R}_{ij}$.
**Step3:** *B2*. Define M rings $\mathcal{E}_k$ with $k=1,2,\ldots,M$.
**Step4:** Put $k=1$.
**Step5:** *B4*. Cover $\mathcal{E}_k$. *B5*. Get $\mathbf{L}_k, \mathbf{N}_k$.
**Step6:** If $k=1$ then put $k=2$, go to **Step5**. If $k \neq 1$ then go to **Step7**.
**Step7:** Check: $C_r \cdot \frac{\mathbf{L}_k}{\mathbf{L}_{k-1}} \leqslant \frac{\mathbf{N}_k}{\mathbf{N}_{k-1}}$,

where $.95 \leqslant C_r \leqslant .98$ is *a corrector*, which is supposed to partially compensate the errors due to truncated rectangles. The corrector is determined experimentally after 2-3 runs of the algorithm.
**Step8:** If **Step7** is false then goto **Step5**, if **Step7** is true, then go to **Step9**.
**Step9:** If $k=M$, then go to **Step10**, otherwise set $k=k+1$ and goto **Step7**.
**Step10:** STOP: The job is done.

Figure 1 illustrates four cases for which algorithm was applied.

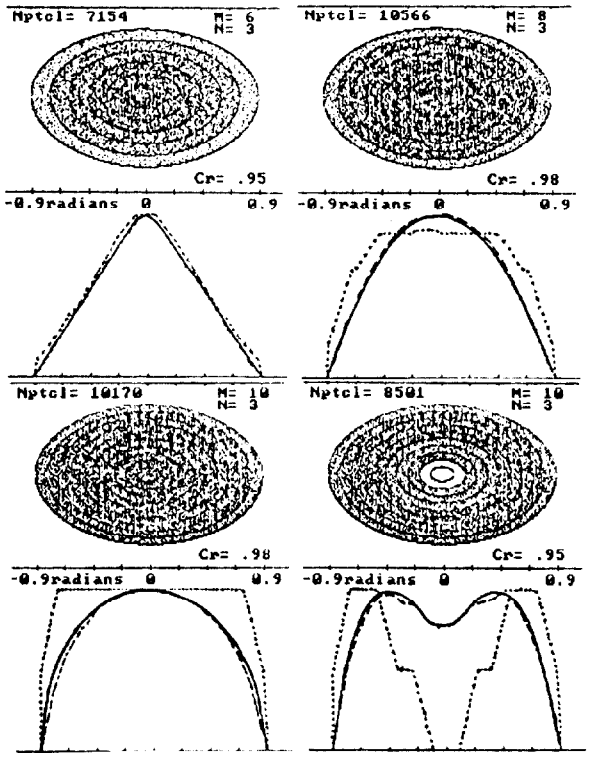


Figure 1. Line density $\lambda$-solid, local density $\rho$-dotted, and line density $\bar{\lambda}$-dashed line.

After all the particles have been deposited in the bunch according to the algorithm, we use the newly created bunch to reconstruct the line density $\bar{\lambda}=\bar{\lambda}(\varphi)$ shown by the dashed line. This gives us an indication of the accuracy of the algorithm. The local density distribution, $\rho$, was computed by direct counting of the particles near the $\varphi$-axis. For stability studies, this $\rho$-distribution needs smoothing treatment.

## IV. Reference

[1] P.W.Krempl, "The Abel-type Integral Transformation with the Kernel $(t^2-x^2)^{-1/2}$ and Its Application to Density Distributions of Particle Beams", CERN MPS/Int. BR/74-1, pp.1-31, March 1974.