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Application of Differential-and-Lie-Algebraic Techniques to the Orbit Dynamics of Cyclotrons

W.G. Davies, S.R. Douglas, G.D. Pusch, and G.E. Lee-Whiting
AECL, Chalk River Laboratories,
Chalk River, Ontario K0J 1J0, Canada.

Abstract — A new orbit-dynamics code, DACYC, is being developed for the TASCC superconducting cyclotron. DACYC makes use of *differential algebra* and *Lie Algebra* to calculate and analyze partial, one-and/or multi-turn maps to very high order. Accurate, three-dimensional, analytic models of the magnetic and *RF* fields are used, which satisfy Maxwell's equations exactly. The maps can be analyzed with *normal-form* methods or to produce linear or high-order phase-space plots.

I. Introduction

Although cyclotrons are only quasi-periodic devices under operational conditions, the use of maps is still a very efficient way of exhibiting and analyzing important orbit-dynamical features, especially resonances. A new orbit dynamics code, DACYC, is being developed for the TASCC [1] superconducting cyclotron. DACYC makes use of the powerful symbiosis of *differential algebra* [2] (*DA*) and *Lie Algebra* [3] to calculate and analyze partial, one-and/or multi-turn maps to very high order. The Lie-algebraic formulation of Hamiltonian mechanics provides powerful methods for representing and analyzing maps, especially the transformation to *normal form*, but requires that the coordinates be canonical. Thus, we integrate Hamilton's equations directly.

The high-order Taylor-series map (the usual aberration expansion) is very efficiently computed from Hamilton's equations with differential algebra, but the Lie map is not. Fortunately, *DA* provides, through its ability to compute high-order derivatives with high accuracy, efficient means for converting the Taylor-series map into a Lie-map when desired. However, for *DA* techniques to be used, the equations of motion and the electromagnetic potentials must be in analytic form.

II. Equations of Motion

The relativistic Hamiltonian in cylindrical coordinates, the "natural" coordinate system for cyclotrons, is [4]

$$H = -\rho A_\theta - \rho\left\{\left(\frac{P_T+\varphi}{c}\right)^2 - m^2c^2 - (\boldsymbol{P}_\perp - \boldsymbol{A}_\perp)^2\right\}^{\frac{1}{2}} \tag{1}$$

where H is the negative of the canonical momentum conjugate to θ, θ is the independent variable, ρ is the radius, P_T is the total energy, φ is the electromagnetic scalar potential, and $\boldsymbol{P}_\perp$ and $\boldsymbol{A}_\perp$ are the transverse components of the momentum and the electromagnetic vector potential, respectively; A_θ is the azimuthal component of the vector potential, m is the particle mass and c is the velocity of light. Hamilton's equations in Poisson-bracket form are:

$$\dot{\boldsymbol{Z}} = -[H, \boldsymbol{Z}], \tag{2}$$

where $\boldsymbol{Z} = (\rho, P_\rho, Z, P_Z, T, P_T)$, are the canonical coordinates, and the dot denotes differentiation with respect to θ. Eq. (2) is expanded in a Taylor series via *DA* [5], and the resulting *DA* "vectors" are integrated numerically by a Bulirsch-Stoer integrator [6]. The result is the reference trajectory in cylindrical coordinates, along with the usual Taylor-series map, or aberration expansion,

$$\zeta_i^{FIN} = R_{ij}\zeta_j^{IN} + T_{ijk}\zeta_j^{IN}\zeta_k^{IN} + \cdots \tag{3}$$

where the elements of $\boldsymbol{\zeta} = (x, p_x, z, p_z, t, p_t)$, are the canonical coordinates relative to the reference trajectory; repeated indices are summed. The superscripts, IN and FIN, refer to the initial and final coordinates connected by the map. The order of the map (3) is determined at run-time. Eq. (3) plus the reference trajectory are sufficient if we are interested only in simple phase-space plots and/or the behavior of the reference trajectory. For the study of non-linear behavior, in particular resonances, we transform to the Lie representation [3].

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III. Magnetic Field Representation

For DA to be used, the vector potential, $\boldsymbol{A}$, must be in analytic form. We have constructed a 3-dimensional analytic model of the magnetic field which satisfies Maxwell's equations exactly. The model is divided into four parts:

- 1. the field from the coils;
- 2. magnetization of the iron poles;
- 3. "fields" from the trim rods;
- 4. residual field (perturbations).

The vector potential of a pair of circular loops of radius a carrying a current I can be written in the form

$$A_\theta(a,b) = a\mu_0 I \int_0^\infty e^{-bs} J_1(\rho s) J_1(as)\cosh(zs)\,ds \quad (4)$$

where $|z| < b$; $2b$ is the spacing between the loops. We integrate (4) over the area of the coils by partitioning the coils and making a moment expansion about the center of each cell. The moments can be obtained by recursion from two elliptic integrals. The preceding method greatly reduces the computing time over a "brute-force" integration of (4). Furthermore, the derivatives of A_θ may also be obtained recursively to any order and "poked" into the appropriate DA vector.

The vector potential from the saturated iron poles can be obtained from a polygonal current-sheet approximation. By Ampère's law, the vector potential of a current loop Γ' is:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi} \oint_{\Gamma'} \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|}\, d\boldsymbol{s}', \quad (5)$$

where $\boldsymbol{r}$ is the field evaluation point, $\boldsymbol{r}'$ is a point on Γ', and $d\boldsymbol{s}'$ is a line-element of Γ' at $\boldsymbol{r}'$. Choosing Γ' to be a planar polygon, and integrating (5) with respect to z' from $-\infty$ to $-b$ and b to ∞ (where b is the gap), we obtain an expression involving only log and arctan, both of which can be differentiated in DA.

Each hill and valley region is represented in this model. The hill edges are "softened" by the addition of line distributions of magnetic dipoles, which help to fit the details of the hill fringe-field region. A least-squares fit of items 1 and 2 of this model to the measured midplane field is shown in fig. 1. The quality of the fit is quite impressive; the RMS error in fig. 1 is already only 0.237%, and we expect this to be reduced even further by additional refinements to our model.

The TASCC cyclotron [1] uses 104 saturated iron "trim-rods" instead of trim-coils to fine-tune the magnetic field.

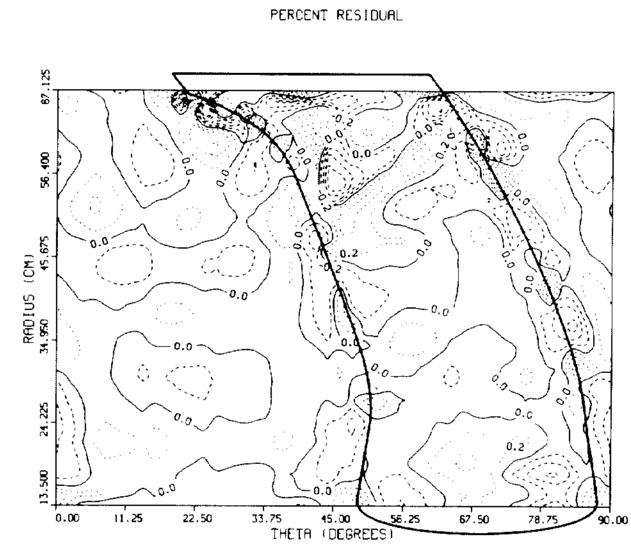

Figure 1: Percent deviation of the fitted magnetic field from the measured mid-plane magnetic field; dotted (dashed) lines denote positive (negative) contours, respectively. Contours are spaced 0.2% apart. The heavy solid line outlines the "hill" boundary.

The potential of each trim-rod pair may be represented either by a current-sheet approximation or a multipole expansion. The most computationally efficient representation has not yet been determined.

The residual field, $\Delta B_z = (B_{meas.} - B_{model})_z$, is the difference between the measured mid-plane field and the model field and includes perturbations from yoke penetrations, extraction elements and their compensation bars. The residual field will be fitted to a Fourier-Bessel series derived from a Hertz potential $\boldsymbol{\Pi}$. For the static case, $\nabla^2\boldsymbol{\Pi} = 0$, $\boldsymbol{A} = \nabla \times \boldsymbol{\Pi}$, and $\boldsymbol{B} = \nabla \times (\nabla \times \boldsymbol{\Pi})$. We can choose a gauge such that $\boldsymbol{\Pi} = \varpi\boldsymbol{k}$, where $\boldsymbol{k}$ is the unit vector in the z direction. This leads to a simple relationship between the mid-plane field and ϖ.

The magnetic field model outlined above provides a full 3-dimensional description of the field that satisfies Maxwell's equations exactly. This includes not only the acceleration region between the poles, but the fringe-field region out to the vicinity of the yoke wall.

IV. RF-Field Representation

Due to peculiarities of the RF cavity design, there are distributed vertical components of the electric field. Thus, the usual kick approximation cannot be used. We use a Fourier-Bessel expansion for the RF field, derived from a dynamic Hertz potential. Because the geometries of the magnetic and RF elements are very similar, the same trigonometric and Bessel functions are used, involving very little additional computation.

V. Lie-Algebraic Analysis

The Taylor series map (3) can be transformed [3] into a product of Lie-maps

$$\mathcal{M} = \mathcal{M}_2\mathcal{M}_3\mathcal{M}_4\cdots \qquad (6)$$

where each of the factors $\mathcal{M}_n$ is generated by an n^{th}-order homogeneous polynomial — called *Lie-polynomials* — in the canonical coordinates ζ; $\mathcal{M}_2$ is equivalent to R, and $\mathcal{M}_3$ to T in (3). The Lie-polynomials provide a much more concise representation of the map than (3); furthermore, the Lie maps in (6) preserve the symplectic nature of $\mathcal{M}$ exactly, whereas the Taylor series (3) does not. With the help of DA, (6) can be transformed into *normal form* [3]

$$\mathcal{M} = \mathcal{A}^{-1}\mathcal{N}\mathcal{A} \qquad (7)$$

where $\mathcal{N}$, the *normal map*, now takes its "simplest" form, containing only *tune-shift* and *resonance* terms. $\mathcal{A}$ is the non-linear canonical transformation connecting $\mathcal{M}$ with $\mathcal{N}$. That we can incorporate all the other non-linearities into $\mathcal{A}$ makes the analysis of the tune-shifts and resonances much easier! The Lie-algebra library DALIE of Forest [7] is used for this analysis.

VI. Example

A simplified model of the magnetic and RF fields, valid only near the inner region of the cyclotron, has been used to study the effects of the vertical components present in the RF field, especially in push-pull, or "π-mode". Calculations of the first 5 turns in the cyclotron (in order to include one complete betatron oscillation in the vertical plane) were performed through second-order aberrations ($\mathcal{M}_3$ or T) for ^{35}Cl at 30 MeV/u. The effect on the reference trajectory is shown in fig. 2. These results are quite sensitive to the details of the RF field, and especially to the RF phase when the beam passes through the carbon stripping foil at injection (see ref. 1, and references therein). From the "first order" map, R, we find that the radial and vertical tunes are $\nu_x = 1.006$, and $\nu_z = 0.227$, in agreement with measurement. In spite of the fact that the vertical RF field components break midplane symmetry and mix the x and z planes, the extra nonlinear couplings do not appear to have deleterious consequences at inner radii. This may not remain true when the beam passes through resonances, and is already known not to be true in the vicinity of the Walkinshaw resonance, which is encountered near the extraction radius.

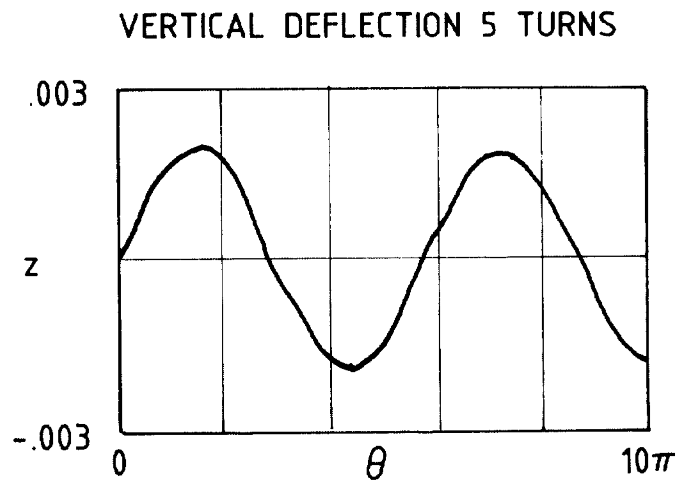

Figure 2: Projection of the reference trajectory on the vertical plane as a function of azimuth. This figure illustrates the effect of the vertical components in the RF field.

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