NONLINEAR DYNAMICS OF ELECTRONS IN ALTERNATING-SIGN TOROIDAL MAGNETIC FIELD

Yu.L. Martirosian, M.L. Petrosian,
Yerevan Physics Institute, Armenia, USSR

At present the possibility of using toroidal magnetic fields for the increase of intensity of accelerated beams is widely discussed in scientific literature on acceleration physics.

In Refs [1,2] it was proposed to use a time constant toroidal magnetic field as a magnetic path of average energy induction accelerator and storing rings, where the vertical drift of particles was suppressed by means of pairs of quadrupole lenses regularly spaced along the torus. The use of an alternating sign toroidal magnetic field as a magnetic path in wide energy range of accelerated electron beams (from several keV to some MeV) was discussed in Refs [3,4]. In such a system the vertical drift of particles in a separate section is compensated by an inverse drift in the neighboring section. The toroidal magnetic field was reported in [5,6] to be applied in a modified betatron in addition to a conventional betatron field at the initial stage of acceleration with the view of essential enhancement of the current of accelerated particles.

In the present work the nonlinear dynamics of electrons in time-constant alternating-sign toroidal magnetic field is considered on the basis of Hamiltonian formalism.

Let N be the number of magnetic path periods each of $2\,l_o = \frac{2\pi R_o}{N}$ length, consisting of sections of toroidal magnetic fields of opposite directions. Here R_o is the major radius of the torus and l_o is the length of one coil. The ends of toroidal solenoids are assumed to be closely situated. Assuming that $l_o \ll R_o$ (N>>1), one can write the vector potential of the magnetic field of such a configuration in cylindrical system of axes (r,θ,z) as [7]

$$A_r(r,\theta,z) = -A_o \frac{z}{[(r-R)^2+z^2]^{1/2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} * \quad (1)$$

$$* I_1\left(\frac{\alpha_k}{l_o}\sqrt{(r-R)^2+z^2}\right) * K_1\left(\alpha_k \frac{\rho_o}{l_o}\right)\cos(2k+1)N\theta$$

$$A_z(r,\theta,z) = -A_o \frac{r-R}{[(r-R)^2+z^2]^{1/2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} *$$

$$* I_1\left(\frac{\alpha_k}{l_o}\sqrt{(r-R)^2+z^2}\right) * K_1\left(\alpha_k \frac{\rho_o}{l_o}\right)\cos(2k+1)N\theta$$

$$A_\theta = 0$$

where $A_o = \frac{Ind}{2}$

I – the current in coil windings
n – the number of loops per unit length
$d = 2\rho_o$ – the minor diameter of torus

$\alpha_k = \pi(2k+1)$

$I_1(z)$ and $K_1(z)$ are modified first-order Bessel functions

In the absence of electric field the Hamiltonian function is known to be

$$H = c\left[(\vec{p} - \frac{e}{c}\vec{A})^2 + m_o^2c^2\right]^{1/2} \quad (2)$$

where $\vec{p}$ – the vector of canonical momentum
m_o – the rest mass of an electron
c – the speed of light
$\vec{A}$ – the time-independent vector potential determined by the formula (1)
e – the charge of an electron

As the Hamiltonian function is not explicitely time-dependent, it is an integral of motion and we can equate it to the total energy of an electron.

After the canonical transformation of variables $(r,\theta,z) \to (\rho,\theta,\varphi)$ (ρ and φ are the polar coordinates in the cross-section of torus)

$$\begin{cases} r - R_o = \rho\cos\varphi \\ z = \rho\sin\varphi \end{cases} \quad (3)$$

$$\begin{cases} P_z = P_\rho\cos\varphi - P_\varphi\sin\varphi \\ P_z = P_\rho\sin\varphi + P_\varphi\cos\varphi \end{cases}$$

one can reduce this problem to an equivalent one with simpler Hamiltonian function

$$H(x_1, x_2, y_1, y_2) = \frac{y_1^2 + y_2^2}{2} + \frac{1}{2}[y_3 - \nu h(x_1,x_2)]^2 \quad (4)$$

where $y_1 = \frac{P_\rho}{P}$; $y_2 = \frac{P_\theta}{P}$; $y_3 = \frac{P_\varphi}{P}$; are new canonical momenta,

$x_1 = \frac{\rho}{l_o}$; $x_2 = \frac{2N\theta}{\pi}$; $x_3 = \varphi$; – are corresponding coordinates,

$$h(x_1,x_2) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)} I_1(\alpha_k x_1) K_1(\alpha_k x_o)\cos(2k+1)x_2$$

$$\nu = \frac{e A_o}{pc};$$

p – the momentum of an electron.

As the new Hamiltonian is independent of $x_3 = \varphi$, the corresponding momentum

0-7803-0135-8/91$01.00 ©IEEE

$$y_3 = \frac{m_0 \rho \dot{\phi} + \nu\, h(x_1,x_2)}{P} = \text{const} \qquad (5)$$

that is an expression of Bush theorem of the conservation of angular momentum. With the initial conditions $\phi_0 = 0$; $x_{20} = \pm 1$ we have $y_3 = 0$ and the Hamiltonian could be written in the standard form with variables "action-angle"

$$H(x_1,x_2,y_1,y_2) = \frac{y_1^2+y_2^2}{2} + \frac{\nu^2}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n} h_k(x_1,x_2) * h_{n-k}(x_1,x_2) \qquad (6)$$

Using the series expansion of Bessel function into the powers of argument [8]

$$I_1(z) = \frac{z}{2}\sum_{m=0}^{\infty}\frac{\left(\frac{z}{2}\right)^{2m}}{m!\,\Gamma\,(m+2)}$$

one can normalize the Hamiltonian by writing it in the following form

$$H = H_0 + H_2 + H_4 + \ldots\ldots \qquad (7)$$

where

$$H_0 = \frac{y_1^2 + y_2^2}{2} \qquad (8)$$

$$H_2 = x_1^2 \frac{\nu^2}{16}\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^n K_1(\alpha_k x_0) K_1(\alpha_{n-k} x_0) C_{nk}(x_2)$$

$$H_4 = x_1^4 \frac{\nu^2}{64}\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^n K_1(\alpha_k x_0) K_1(\alpha_{n-k} x_0) C_{nk}(x_2) * \alpha_{n-k}^2$$

$$x_0 = \frac{\rho_0}{l_0}$$

$$C_{nk} = \cos\,(2k+1)\,x_2\,\cos\,[2(n-k)+1]\,x_2$$

The allowance in the expression (7) for terms up to the quadratic Hamiltonian leads us from canonical equations

$$\frac{dx_i}{d\tau} = \frac{\partial H}{\partial y_i}; \qquad \frac{dy_i}{d\tau} = -\frac{\partial H}{\partial x_i}; \qquad i = 1,\ 2 \qquad (9)$$

to Hill equations with respect to x_1

$$\ddot{x}_1 + \nu^2[a_0 + \sum_{m=1}^{\infty} a_m(x_0)\cos 2mx_2]\, x_1 = 0$$

$$x_2 = y_{20}\tau + x_{20} \qquad (10)$$

where the top dot denotes the differentiation with respect to dimensionless variable τ, the solution of which is obtained with the help of Flokier functions.

The points of unstable motion in the linear approximation [4] are in reality narrow strips which begin in the points

$$2\nu^2 a_0 = k^2; \qquad k = 1,\ 2,\ 3, \ldots. \qquad (11)$$

on the positive semi axes a_0.

As it seen from (7) $a_0 > a_1$ and the motion is almost generally stable. However, already for $x_0 = 2$ (i.e. the length of coil becomes equal to its diameter) $a_0 \approx a_1$ and the "islets" of unstable motion enlarge and we approach the instability range.

Solving the equation (10) by means of Hill or averaging method, one can obtain highly accurate solutions of canonical equations (9) after an appropriate canonical transformation using the subsequent term of expansion (7). This transformations and final results are rather cumbersome, and it is difficult to estimate and check them. However, it is possible to draw conclusions about the solutions (9) even without the solution of these equations.

As in the normalized Hamiltonian (7) the quadratic part is a function of fixed sign, then taking it as the Lyapunov function, we can conclude on the basis of Mozer theorem [9] that the stability conditions

$$2\nu^2 a_0 \neq k^2; \qquad k = 1,\ 2,\ 3, \ldots.. \qquad (12)$$

$$a_0 > a_1$$

are sufficient for the existence of solutions of exact equations, which are stable in the sense of Lyapunov stability.

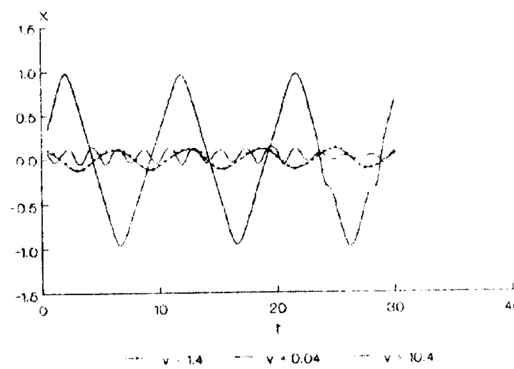

Fig.1. Electron radial oscillation trajectories in alternating-sign toroidal magnetic fields. All parameters are given in the text.

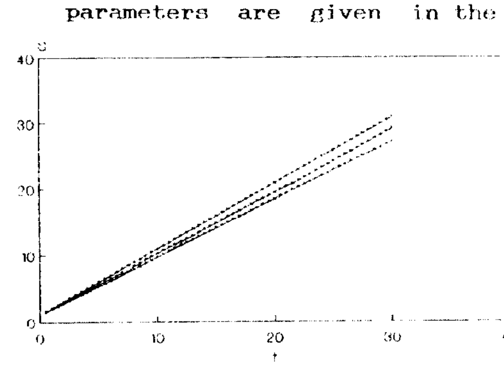

Fig.2. Electron longituninal oscillation trajectories in alternating-sign toroidal magnetic fields. All parameters are given in the text.

Shown in Fig.1 are the plots of radial oscillations (vertical oscillations differ from the radial ones by the phase shift of $\pi / 2$) versus the dimensionless time τ, which were obtained on a computer for the values of characteristic parameter ν = 0.04; 0.4; 10.4. In Fig. 2 the dependence of longitudinal oscillations on τ is given for the same values of parameters. For the magnetic field intensity in the center of the coil H_0=1500 Oersted (for warm magnets), R_0= 150 cm, d = 4 cm, N = 30 (the length of the coil is 15 cm), the kinetic energy of $E_k \approx 6$ MeV corresponds to ν = 0.04, and $E_k \approx 10$ kev corresponds to ν = 10.4. In all the cases the values of initial coordinates and velocities are $r_0 - R_0$= 0.1 cm; z_0=0.1 cm $v_{or} \approx v_{oz} \approx 0.01\, v_{o\vartheta}$.

As it seen from Fig.1 and Fig.2, when the electron energy grows from 10 keV to nearly 1 MeV (to relativistic velocities), then the compression of ellipses in momenta takes place, and with further increase in energy in relativistic range the extension of ellipses in directions of both axes occurs.

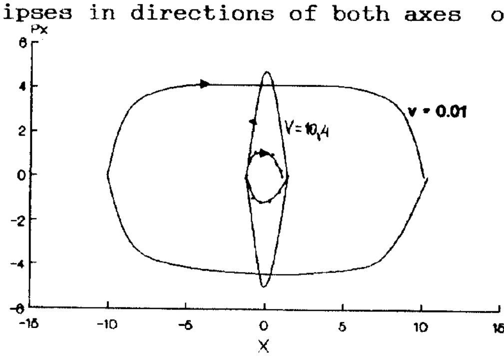

Fig.3. Electron phase trajectories in alternating-sign toroidal magnetic fields. All parameters are given in the text.

We see in Fig.2, that small amplitude periodical oscillations overlap the monotonous growth of longitudinal motion. The amplitude of these oscillations increase with the growth of initial transverse velocities to values $V_{tr}/ V_{lon} \approx 1 / \nu$ and the edging transient field of coils may serve as a magnetic trap.

The restrictions on maximum attainable electron energy stem from the vertical drift within one element of periodicity. At the application of superconducting solenoids one can increase the magnetic field intensity by more than one order of magnitude and hence essentially increase the energy of accelerated beams.

In Fig.3 the phase patterns of transverse oscillations are given for the same values of the system parameters.

References

[1] Yu.L. Martirosian and M.L. Petrosian, "The motion of electrons in the azimuth-circular magnetic field," ЕФИ-414/21/80 (in Russian).

[2] M.A. Akopov, Yu.L. Martirosian and M.L. Petrosian, A.C. N 768378, БИ. 1982, 41, p. 275. (in Russian).

[3] Ya.S. Derbenev, Yu.L. Martirosian and M.L. Petrosian, ЖТФ, v.59, 8, 1989. (in Russian).

[4] Yu.L. Martirosian, ЖТФ, v.60, 8, 1990. (in Russian).

[5] N. Rostoker, Bull. APS. vol.25, p. 854, 1980.

[6] G. Barak et al, IEEE Trans. Nucl. Sci. vol. NS-28, 3, pp.792-798, 1981.

[7] H. Buchholz, Electrische und Magnetische Potentialfelder, Springer - Verlag, 1957.

[8] M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions, National Bureau of Standards, 1964.

[9] G.E.O. Giacaglia, Perturbation Methods in Non-Linear Systems, New York, 1972.