Relativistic Mulipactoring

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Increasingly, since multipactoring was first described by Philo Farnsworth in the 1930's, [1] it has been the supposed defect in many microwave devices. It is thought to have occurred in proton linear accelerators (between drift tubes), klystron cavities, obstacle loaded waveguides and super-conducting cavities. In many such devices field strengths are often rather high so that the classical calculation [2] is presumably not adequate. This report presents a relativistic calculation of the multipactoring condition for the two surfaces phenomenon.

Secondary election multiplier action (multipactoring) may exist between two opposed surfaces when a cyclic voltage is maintained between them. If a primary electron impinges on one of the opposed surfaces, causing the emission of more than one secondary electron, a sustained exchange of charge will occur, if at about the instant of emission the applied electric field reverses and accelerates the secondaries to the opposing surface, where a similar circumstance occurs. Once initiated, for whatever reason, a multipactor "discharge" then consists of a thin electron cloud that is driven back-and-forth between the opposed surfaces (or gap) in response to the applied field. The charge exchange in each half-cycle will increase to a limit set by the perveance of the gap, reaching steady state in a number of half-cycles set by the secondary emission coefficient (SEC). This perveance limit may be interpreted that, as the electron density increases mutual repulsion causes some electrons to fall outside the phase range over which multipactoring can occur, thereby limiting the maximum electron density in the exchange cloud.

The appropriate condition necessary to produce multipactoring may be derived from the electronic equation of motion,

$$\frac{d(\gamma\dot{z})}{dt} = \frac{eV_0}{md}\sin(\omega t+\varphi) \qquad (1)$$

where $E = (V_0/d)\sin\omega t$ is the cyclic field across the gap(d), Vo being the peak gap voltage and φ the field phase at emission.

Integrating, with dz/dt=0, t=0;

$$\gamma\frac{dz}{dt} = \frac{eV_0}{\omega md}\left(\cos\varphi-\cos(\omega t+\varphi)\right) \qquad (2)$$

Noting that, for the one-dimensional (linear trajectory) case,

$$\gamma = \frac{1}{\sqrt{1-\left(\frac{1}{c}\frac{dz}{dt}\right)^2}} \qquad (3)$$

and eq(2) becomes

$$\frac{dz}{dt} = \frac{\frac{eV_0}{\omega md}\left[\cos\varphi-\cos(\omega t+\varphi)\right]}{\sqrt{1+\left(\frac{eV_0}{\omega mdc}\right)^2\left[\cos\varphi-\cos(\omega t+\varphi)\right]^2}} \qquad (4)$$

With the transformation

$$x = \left(\frac{eV_0}{\omega md}\right)\left[\cos\varphi-\cos(\omega t+\varphi)\right] \qquad (5)$$

$$\frac{dx}{dt} = \left(\frac{eV_0}{\omega md}\right)\sqrt{\sin^2\varphi+\left(\frac{2\cos\varphi}{eV_0/\omega md}\right)x-\left(\frac{x}{eV_0/\omega md}\right)^2}$$

eq(4) reduces to

$$\left(\frac{eV_0}{\omega d}\right)dz = \frac{dx}{\left[\sin^2\varphi+\left(\frac{2\cos\varphi}{eV_0/\omega md}\right)x+\left(\frac{x\sin\varphi}{c}\right)^2-\left(\frac{x}{eV_0/\omega md}\right)^2+\frac{2x^3\cos\varphi}{eV_0/\omega mdc^2}-\left(\frac{x^2}{eV_0/\omega md}\right)^2\right]^{1/2}} \qquad (6)$$

which may be directly integrated (See Grobner and Hofreiter, Integraltafel, Unbestimmte Integrale, Wien (1957) no. 867 sec 244 p 88) but the unravelment of the result is too complicated for practical use, so that the next best procedure seems to be an expansion of the denominator of eq (4) in a binomial series, followed with term-by-term integration. Then,

$$\frac{dz}{dt} = \left(\frac{eV_0}{\omega md}\right)\left[\cos\varphi-\cos(\omega t+\varphi)\right]\cdot\left[1-\frac{1}{2}\left(\frac{eV_0}{\omega mdc}\right)^2\left[\cos\varphi-\cos(\omega t+\varphi)\right]^2\right] \qquad (7)$$

Integrating, with Z=0, ωt=0;

$$\begin{aligned} z = {} & \frac{eV_0}{\omega^2 md}\left(\omega t\cos\varphi-\sin(\omega t+\varphi)+\sin\varphi\right) \\ & -\frac{1}{2}\frac{eV_0}{\omega^2 md}\left(\frac{eV_0}{\omega mdc}\right)^2\Big[\omega t\cos^3\varphi-3\cos^2\varphi \\ & \left(\sin(\omega t+\varphi)-\sin\varphi\right)+\frac{3}{2}\cos\varphi \qquad (8) \\ & \left(\sin(\omega t+\varphi)\cos(\omega t+\varphi)-\sin\varphi\cos\varphi \right. \\ & \left. +\,\omega t\right)-\frac{1}{3}\sin(\omega t+\varphi) \\ & \left(\cos^2(\omega t+\varphi)+2\right)+\frac{1}{3}\sin\varphi\left(\cos^2\varphi+2\right)\Big] \end{aligned}$$

Since $\omega t=(2n+1)\pi$ to achieve the multipactor condition, that the transit time be an odd number of half cycles, eq(8) becomes

$$z = \frac{eV_0}{\omega^2 md}\Big[(2n+1)\pi\cos\varphi + 2\sin\varphi\Big]$$

$$-\frac{1}{2}\frac{eV_0}{\omega^2 md}\left(\frac{eV_0}{\omega mdc}\right)^2\Big[(2n+1)\pi \cos^3\varphi + 6\cos^2\varphi\sin\varphi + \frac{3}{2}\cos\varphi\big((2n+1)\pi - 2\sin\varphi\cos\varphi\big) + \frac{2}{3}\sin\varphi\left(\cos^2\varphi+2\right)\Big] + \cdots \qquad (9)$$

Putting $z=d$, (the gap separation) and solving the resulting quadratic;

$$\left(\frac{d}{\lambda/2}\right)^2 = \frac{eV_0}{2\pi mc^2}\left(1+\sqrt{1-\frac{eV_0}{mc^2}\frac{5}{\pi}}\right) \qquad (10)$$

For $eV_o/mc^2 <<1$. $\left(\frac{d}{\lambda/2}\right)^2 = eV_o/\pi mc^2$, which is the classical rule; eq(10) is limited to V_o=320KV by the truncation in eq(7). Including further terms in eq(7) would have resulted in a higher order algebraic equation in the solution of eq(10).

The simplicity of the dynamic problem is now further complicated by the secondary emission process. (3). When the primary electron strikes the opposing surface there are a number of competing processes by which it may become depleted of its incident energy. For a sufficiently high energy the electron will loose energy principally by radiation and forward scattering deeper into the target material, although some secondaries are produced at any energy. Eq(7) may be used to determine the impact energy (V) when $\omega t=\pi$; since

$$\gamma = \frac{1}{\sqrt{1-\left(\frac{1}{c}\frac{dz}{dt}\right)^2}} = 1+\frac{eV}{mc^2} \qquad (11)$$

by eq(7)

$$\frac{eV}{mc^2} = \sqrt{1+\left(\frac{eV_0}{\omega mdc}\right)^2\Big[\cos\varphi - \cos(\omega t+\varphi)\Big]^2} - 1$$

$$\doteq \frac{1}{2}\left(\frac{eV_0}{\omega mdc}\right)^2\Big[\cos\varphi - \cos(\omega t+\varphi)\Big]^2$$

$$\doteq 2\left(\frac{eV_0}{\omega mdc}\right)^2\cos^2\varphi \qquad (\omega t=\pi)$$

or,

$$\frac{V}{V_0} = 2\,\frac{eV_0}{mc^2}\left(\frac{\lambda/2}{d\pi}\cos\varphi\right)^2 \qquad (12)$$

For the special case $\varphi=0$ and $eV_o/mc^2 <<1$, substituting from eq(10), the gap spacing condition;

$$\frac{V}{V_0} = \frac{2}{\pi}$$

Quite obviously there is a phase span over which multipactoring may occur. Classically $0<\varphi<\arctan 2/(2n+1)\pi$. But, in the present case the method of solution prevents obtaining an estimate of the phase range.

It may also be noticed that the equation of motion, eq(1), is simplistic in that reradiation by the accelerated electron beam is ignored.(4)

Alternatively, if we assume

$$\left(\frac{eV_0}{\omega mdc}\right)^2\Big[\cos\varphi - \cos(\omega t+\varphi)\Big]^2 << 1$$

as will usually be the case, eq(4) reduces to the classical case, the solution of which has been considered elsewhere (ref 2).

Such complications as eq(4) are common in relativistic dynamics, to the chagrin of analysts, but can be soved for special cases by means of computing machines, however unstatisfying that may be.

Literature Cited

(1) P. Farnsworth, Jour. Franklin Inst. 2,411(1934)

(2) See, for example, W.J. Gallagher IEEE TRrans Nuc.Sci. NS-26(3), 4280 (1979)

(3) S.R. Farrell, et al. IEEE Trans. Nuc.Sci. NS-32(5), 2900 (1985)
E. Baroody, Phys. Rev. 78,780 (1950)

(4) For a discussion of radiation damping see L. Landau and E. Lifshitz, The Classical Theory of Fields. Cambridge, USA (1951) ch9, p210 W.K.H. Panofsky and M. Phillips, Classical Electricity and Magnetism. Cambridge, USA(1955) ch 19, p297.