Stratonovich expansion and beam-beam interaction

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Abstract A general prescription is proposed for the study of coherent phenomena in electron storage rings due to a localized nonlinear force. The prescription is based on expanding the distribution function into a series using generalized Hermite polynomials in two dimensions (Stratonovich expansion). When the series is terminated at the lowest order, it gives the Gaussian approximation. The prescription is applied to strong-strong and weak-strong beam-beam interactions in e^+e^- colliding storage rings at the next-to-lowest order approximation.

1 Introduction

Recently, the present author proposed a solvable model[1] of the beam-beam interaction based on the Gaussian approximation of the distribution functions. It illustrated some of the characteristic features of the problem qualitatively well. Quantitatively, however, there were some disagreements. This seems to come from the lack of degrees of freedom of the model, since we represented the distribution functions by only three moments.

We, here, will try one possible method based on an expansion of the distribution function into infinite series and truncating it at a finite order. In the next-to-lowest order approximation, the model presents improved quantitative agreement with the multiparticle tracking. Mainly, this paper is a review of Ref.[2] but contains some refinement.

2 Stratonovich expansion

As canonical variables in the 2-dimensional phase-space, we use

$$X^1 = \frac{x}{\sqrt{\beta}}, \quad X^2 = \frac{\alpha x + \beta x'}{\sqrt{\beta}},$$

where x and x' are transverse coordinate and its slope, and α and β are Twiss parameters.

Stratonovich expansion[3], here, is an expansion of two dimensional distribution function $\psi(X^1, X^2)$ around the two dimensional Gaussian distribution,

$$G(\vec{X};g) = \frac{1}{2\pi\sqrt{\det g}} \exp -\phi,$$

$$\phi = \frac{1}{2} g_{\alpha\beta} X^\alpha X^\beta,$$

where $g_{\alpha\beta}$ is the inverse of $g^{\alpha\beta}$,

$$g^{\alpha\beta} = < X^\alpha X^\beta >,$$

and $\det g = g^{11}g^{22} - (g^{12})^2$. Here and in what follows, we employ **Einstein's summation convention**; when the same symbol appears in both upper and lower indices simultaneously, a summation with respect to the symbol from 1 to 2 is implied. We start from the following fact. *Any distribution function $\psi(\vec{X})$, which is symmetric in phase-space $[\psi(-\vec{X}) = \psi(\vec{X})]$, which is normalized to unity, and which falls exponentially at infinity, can be expanded as*

$$\psi(\vec{X}) = G(\vec{X};g)P(\vec{X};g,Q),$$

$$P(\vec{X};g,Q) = 1 + \sum_{\substack{n=4 \\ even}} \frac{1}{n!} Q^{\alpha_1\alpha_2\cdots\alpha_n} H_{\alpha_1\alpha_2\cdots\alpha_n}(\vec{X};g). \qquad (1)$$

Here the sum extends over all even numbers from 4 to infinity. Here, H is the generalized Hermite polynomial,

$$H_{\alpha_1\alpha_2\cdots\alpha_n}(\vec{X};g) = e^{\phi} \prod_{i=1}^{n} (-\partial_{\alpha_i}) e^{-\phi}.$$

The Q's are called quasi-moments. *For given ψ, the quasi-moments are obtained as*

$$Q^{\alpha_1\alpha_2\cdots\alpha_n} = < H^{\alpha_1\alpha_2\cdots\alpha_n}(\vec{X};g) >,$$

where $< \ >$ is the expectation value with respect to ψ and

$$H^{\alpha_1\alpha_2\cdots\alpha_n} = g^{\alpha_1\beta_1} g^{\alpha_2\beta_2} \cdots g^{\alpha_n\beta_n} H_{\beta_1\beta_2\cdots\beta_n}.$$

Note that $Q^{\alpha\beta} = Q_{\alpha\beta} = 0$ by definition.

In this paper, we truncate Eq.(1) at $n = 4$. Thus, the distribution function ψ is represented by 8 parameters,

$$(g^{\alpha\beta}, Q^{\alpha_1\alpha_2\alpha_3\alpha_4}).$$

3 Weak-Strong Model

For the sake of simplicity, we start from the weak-strong case. We track the changes of g and Q around a ring. A ring is composed of one interaction point (IP) and an arc.

The betatron oscillation with the radiation effect is represented by the following mapping[4]:

$$g^{\alpha\beta}_{new} = \lambda^2 U^\alpha{}_{\alpha'} U^\beta{}_{\beta'} g^{\alpha'\beta'}_{old} + (1-\lambda^2)\varepsilon\delta^{\alpha\beta},$$

$$Q^{\alpha_1\alpha_2\cdots\alpha_n}_{new} = \lambda^n U^{\alpha_1}{}_{\beta_1} U^{\alpha_2}{}_{\beta_2} \cdots U^{\alpha_n}{}_{\beta_n} Q^{\beta_1\beta_2\cdots\beta_n}_{old},$$

$$U^\alpha{}_\beta = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix},$$

where μ is the unperturbed phase advance during the arc.

$$\lambda = \exp(-1/T_\varepsilon), \quad T_\varepsilon = \frac{\text{transverse damping time}}{\text{flight time during the arc}},$$

and ε is the nominal emittance (i.e., without beam-beam effect).

The beam-beam kick, at the IP, is represented by the following:

$$g^{\alpha\beta}_{new} = < (X+F)^\alpha (X+F)^\beta > . \qquad (2)$$

$$Q^{\alpha_1\alpha_2\cdots\alpha_n}_{new} = < H^{\alpha_1\alpha_2\cdots\alpha_n}(\vec{X}+\vec{F}; g_{new}) > . \qquad (3)$$

Here, $\vec{F} = (0, F)$ is the beam-beam kick

$$F(X) = 8\pi\varepsilon\eta \frac{1}{X} [\exp(-\frac{X^2}{2g^{11}_*}) - 1],$$

where η is the nominal beam-beam parameter and $g^{11}_* = \varepsilon$ is the g^{11} of the strong beam. To evaluate the r.h.s. of Eqs.(2) and (3), we use ψ_{old}.

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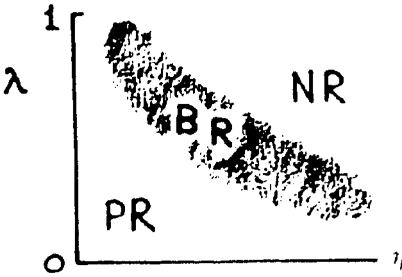

Figure 1: An illustration of regions **PR**,**BR** and **NR**.

4 Intrinsic Singularity

As in the case of the Gaussian approximation, it is expected that the system falls into a period one fixed point in (g, Q) space.

We can, however, show that there cannot be a steady state in some cases. Let us first "assume" that (g, Q) is already settled in the steady state (g^∞, Q^∞), which refers to (g, Q) just before the IP. It is useful to use the following vectors:

$$\vec{Q} \equiv \begin{pmatrix} Q^{1111} \\ Q^{1112} \\ Q^{1122} \\ Q^{1222} \\ Q^{2222} \end{pmatrix}, \quad \vec{H} \equiv \begin{pmatrix} H_{1111} \\ H_{1112} \\ H_{1122} \\ H_{1222} \\ H_{2222} \end{pmatrix}.$$

Now, we can show that

$$\vec{Q}^\infty = \frac{\lambda^4 O}{1 - \lambda^4 O S(g_\infty)} \{\vec{R}(g_\infty) - \vec{Z}(g'_\infty)\}. \quad (4)$$

Here,

$$\vec{Z}(g) = \begin{pmatrix} 3(g^{11})^2 \\ 3g^{11}g^{12} \\ g^{11}g^{22} + 2(g^{12})^2 \\ 3g^{22}g^{12} \\ 3(g^{22})^2 \end{pmatrix},$$

g' is g just after the beam-beam kick, $\vec{R}$ is 5-vector defined by $R_n = I(4, n, 0, 0)$, S is the $(5,5)$ matrix defined by $S_{n,m} = I(4, n, 4, m)$ (n and m run from 0 to 4), where

$$I[N, n, M, m] = \frac{1}{m!(M-m)!} < X^{N-n}\{P + F(X)\}^n \vec{H}[M]_m >_0,$$

$[< \ >_0$ is the average with respect to $G(\vec{X}, g)]$ and O is a $(5,5)$ matrix

$$O_{k,l} = \sum_{\Omega} \begin{pmatrix} 4-k \\ a \end{pmatrix} \begin{pmatrix} k \\ b \end{pmatrix} (-1)^{k-b} \cos^{4-k-b-a}\mu \sin^{k-b+a}\mu,$$

where the sum extends over $\Omega = \{(a, b); a + b = l, \ 0 \le a \le 4 - k, \ 0 \le b \le k\}$.

From Eq.(4), it is clear that Q becomes infinite in the region of (λ, η), where (one of the eigenvalues of) the denominator vanishes. The situation may roughly be illustrated as in Fig.1. We divide the region as follows:

perturbative region (PR): the region around the origin.

border region (BR): the neighboring region of the curve where the denominator of Eq.(4) vanishes.

nonperturbative region (NR): the region far from the perturbative region.

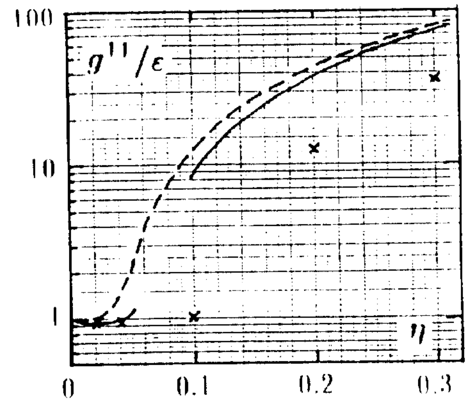

Figure 2: The beam size (g^{11}/ε) vs η in weak-strong case.

5 Features of the Model

To track the change of (g, Q) turn by turn, we choose, as the initial values, $(g_G, 0)$, which is the solution of the Gaussian approximation adopted in Ref.[1]. For some parameters, presumably corresponding to the **BR**, the system shows strange behavior and provides nothing physical. Otherwise, it falls into a period one fixed point. In Fig.2, we compared the result of the model (solid line) with those of a multiparticle tracking ($\times$'s) and the Gaussian approximation (dashed line). Parameters: $T_\epsilon = 142.8$ and tune is 0.15. The solid line is absent for some domain, which corresponds to the **BR**. The agreement is improved compared to the Gaussian approximation. Another interesting quantity is the normalized excess $E \equiv Q^{1111}/(g^{11})^2$, which is dimension-less and represents a deviation of ψ from a Gaussian. It is observed that $E < 0$ and $|E|$ increases rapidly as η increases from 0. After the **BR**, however, it decreases a little and becomes a constant (almost 0.75).

6 Strong-Strong Case

The method thus far stated can easily be applied to this case in a straightforward manner if we employ the same F: i.e., if we ignore the contribution from Q.

We obtain similar results. In Fig.3, we compare the present model with the Gaussian approximation and the multiparticle tracking. Here,

$$R \equiv g^{11}(e^- \text{ beam})/g^{11}(e^+ \text{ beam}).$$

We assume that the two beams are completely symmetric. We observe that

1. The agreement with the multiparticle tracking is improved.

2. The spontaneous symmetry breaking exists also in the present case as in the Gaussian approximation[1].

3. The **BR** exists also. It is represented by the absence of the solid line in Fig.3 (b).

4. In addition, the excess E of the blown-up beam can be shown to increase in the beginning but decrease after the **BR** and saturate eventually. The excess of the not-blown-up beam decreases after the **BR** and does not saturate.

In the case where the two beams are not symmetric, we observe the flip-flop hysteresis as in the case of the Gaussian approximation[1].

7 Discussion

The border region The most characteristic point of the weak-strong case is the rapid increase of the beam size at some η. The region of this rapid increase seems to be related to the **BR**. Also, in the strong-strong case, the bifurcation point is in it. The **BR** seems to be related to the heart of the beam-beam interaction. The **BR** is due to the truncation of the Stratonovich expansion. It exists even if we truncate the expansion at higher orders. This implies that, to understand the most characteristic points of the beam-beam interaction, we should not truncate it and should use ψ itself. The **BR** is the place where the infiniteness of the degrees of freedom of ψ manifests itself.

Outside the **BR**, the Stratonovich expansion gives reasonable approximation and the numerical agreement seems to be improved more and more when higher and higher order quasi moments are introduced.

Positive definiteness of ψ An unsatisfactory feature of the Stratonovich expansion is the fact that, when truncated, ψ can become negative in some region of phase space. Fortunately, in the present problem, The positive definiteness is broken only slightly in the tail. This problem can be fatal if Q is large but g is small. This in not the case in the present model.

Luminosity reduction by deviation from a Gaussian As stated before, E becomes negative under the beam-beam interaction. It can be shown that the luminosity is smaller for a beam with negative E than for the Gaussian beam with the same g^{11}. It is natural since the density function becomes more flat if it has negative E. Thus, when η is large, the actual luminosity is smaller than that estimated only from g^{11}'s. The opposite can occur when E is positive. For the radiation effect in the final quadrupole magnet in the TeV linear colliders[5], we can expect the opposite[6].

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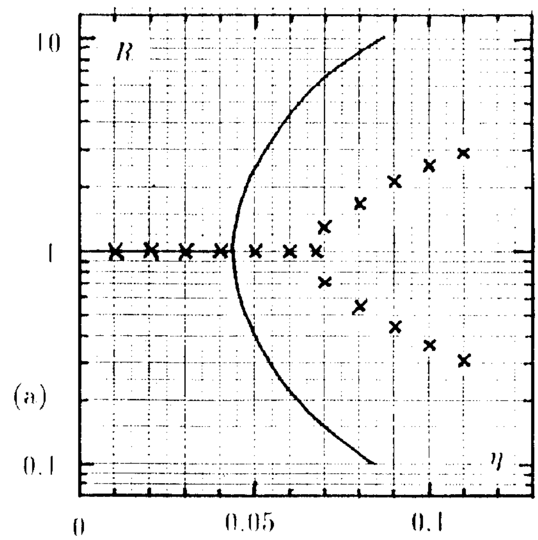

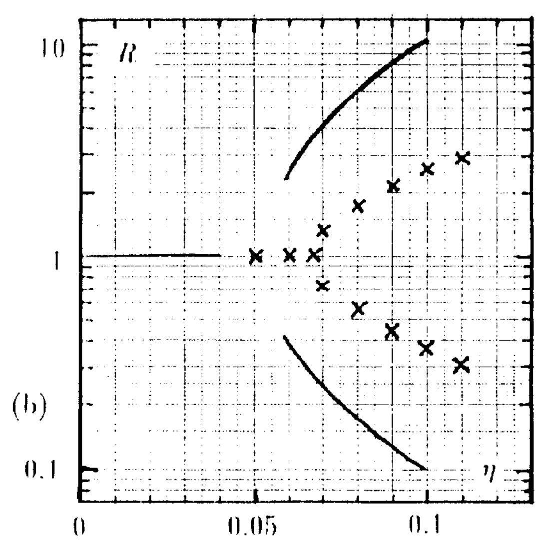

Figure 3: The R v.s. η. (a) Results of the Gaussian approximation. (b) The present model.

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