GENERAL ANALYSIS OF BEAM POSITION MONITORS*

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Abstract

A large proportion of devices used to interact with charged-particle beams in accelerator or storage rings can be classified as pick-ups or kickers. These devices extract information about the particle motion or affect a change in the motion. One device used frequently as pick-up or kicker is made with two little plates with one or more terminations per plate. In this paper the structure with one termination per plate is examined.

1. Charge and Current Induced on the Plates

We consider a bunch of charged particles travelling inside a circular accelerator vacuum chamber. Assuming the radius R of the closed orbit to be much larger than the radius of the vacuum chamber we can treat the particles as travelling along a straight cylindrical pipe of radius b.

Let z be the axis of the pipe and (r,θ) the transverse coordinates. We can associate to the beam a charge and current distribution

$$\rho = Ne\,\frac{\delta(r - r_0)}{r}\,\delta(\theta - \theta_0)f(u) \quad \underline{J} = (0,0,\beta c\rho) \qquad (1)$$

where

$v = \beta c$ is the beam velocity;
N is the number of particles in the beam;
e is the particle charge;
(r_0,θ_0) is the beam position in the transverse plane;
$f(u) = f(z - vt)$ is a function depending on the bunch shape.

The potential due to charge and current distribution is

$$\underline{A} = [0,0,\beta\ V(r,\theta,u)] \qquad (2)$$

Equation (2) allows us to solve the problem through the only scalar Helmotz equation written in cylindrical coordinates for the single azimuthal harmonic $\tilde{V}_m$

$$\frac{d^2\tilde{V}_m}{dr^2} + \frac{1}{r}\frac{d\tilde{V}_m}{dr} - \left(\frac{m^2}{r^2} + q^2\right)\tilde{V}_m = -4Ne\tilde{f}(k)\,\frac{\delta(r - r_0)}{r} \qquad (3)$$

where $q = k/\gamma$ and the symbol $\tilde{\ }$ means fourier transform in the k-space.

Equation (3) is an inhomogeneous Bessel equation with general solution

$$\tilde{V}_m = AI_m(qr) + BK_m(qr) + C_m \qquad (4)$$

where I_m and K_m are modified Bessel functions.

The particular integral C_m is found to be

$$C_m = -4Ne\tilde{f}(k)\,\{I_m(qr_0)K_m(qr) - K_m(qr_0)I_m(qr)\}\,U(r_0 - r) \qquad (5)$$

where U(x) is the Heaveside function.

By imposing the boundary conditions $\tilde{V}_m = 0$ at $r = b$ and that $\tilde{V}_m$ is finite at $r = 0$ we get for the harmonic m of the potential in the region $r_0 < r \le b$

$$\tilde{V}_m = -4Ne\tilde{f}(k)\,\frac{I_m(qr_0)}{I_m(qb)}\,\{I_m(qr)K_m(qb) - I_m(qb)K_m(qr)\} \qquad (6)$$

The surface charge density induced on the wall

$$\sigma(\theta,u) = \sum_{m=0}^{+\infty}\int \varepsilon_m\tilde{\sigma}_m(k)\cos m(\theta - \theta_0)e^{jku}dk \qquad (7)$$

from the Gauss law

$$\tilde{\sigma}_m = \frac{1}{4\pi}\left.\frac{\partial\tilde{V}_m}{\partial r}\right]_{r=b} = -\frac{Ne}{\pi b}\,\tilde{f}(k)\,\frac{I_m(qr_0)}{I_m(qb)} \qquad (8)$$

The surface current density induced on the wall is

$$J_S(\theta,u) = \sum_{m=0}^{+\infty}\int \varepsilon_m\tilde{J}_m(k)\cos m(\theta - \theta_0)e^{jku}dk \qquad (9)$$

where

$$\tilde{J}_m = \beta c\tilde{\sigma}_m \qquad (10)$$

2. The Plate Equations

Scalar and longitudinal vector potentials produced over the plate are derived from the Maxwell equations, assuming that the plate is perfectly conductive,[1]

$$\underline{E}_{tg} = \left.-\frac{1}{c}\frac{\partial\underline{A}}{\partial t} - \nabla V\right]_{tg} = 0 \qquad (11)$$

$$\nabla\cdot\underline{A} + \frac{1}{c}\frac{\partial V}{\partial t} = 0 \qquad (12)$$

The expansion of the potentials in even and odd harmonics gives

$$\tilde{V} = \sum_{m=0}^{\infty}\varepsilon_m\left[\bar{V}_m\cos mg\theta + \bar{\bar{V}}_m\sin\left(\frac{2m+1}{2}\,g\theta\right)\right] \qquad (13)$$

$$\tilde{A}_z = \sum_{m=0}^{\infty}\varepsilon_m\left[\bar{A}_{zm}\cos mg\theta + \bar{\bar{A}}_{zm}\sin\left(\frac{2m+1}{2}\,g\theta\right)\right] \qquad (14)$$

$$\tilde{A}_\theta = \sum_{m=0}^{\infty}\varepsilon_m\left[\bar{A}_{\theta m}\sin mg\theta + \bar{\bar{A}}_{\theta m}\cos\left(\frac{2m+1}{2}\,g\theta\right)\right] \qquad (15)$$

where $g = 2\pi/\varphi_0$.

Equations (11-12) when we take into account the expansions (13-21) give, in the frequency domain, the solutions.

$$\bar{V}_m = \bar{a}_m e^{-j\bar{\beta}z} + \bar{b}_m e^{j\bar{\beta}z} \qquad (16)$$

*Work performed under the auspices of the U.S. Department of Energy

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$$\overline{A}_{zm} = -\frac{\overline{p}}{k_0}\left\{\overline{a}_m e^{-j\overline{p}z} - \overline{b}_m e^{j\overline{p}z}\right\} \quad (17)$$

$$\overline{A}_{\theta m} = j\frac{mg}{k_0 b}\left\{\overline{a}_m e^{-j\overline{p}z} - \overline{b}_m e^{j\overline{p}z}\right\} \quad (18)$$

and

$$\overline{\overline{V}}_m = \overline{\overline{a}}_m e^{-j\overline{\overline{p}}z} + \overline{\overline{b}}_m e^{j\overline{\overline{p}}z} \quad (19)$$

$$\overline{\overline{A}}_{zm} = -\frac{\overline{\overline{p}}}{k_0}\left\{\overline{\overline{a}}_m e^{-j\overline{\overline{p}}z} - \overline{\overline{b}}_m e^{j\overline{\overline{p}}z}\right\} \quad (20)$$

$$\overline{\overline{A}}_{\theta m} = -j\frac{2m+1}{2k_0 b}\, g\left\{\overline{\overline{a}}_m e^{-j\overline{\overline{p}}z} + \overline{\overline{b}}_m e^{j\overline{\overline{p}}z}\right\} \quad (21)$$

Equation (12) is valid in the case that charges and currents are conserved. When we suppose to have current "sources" and/or "losses" in a generic point (z_p, θ_p) on the electrode then equation (13) modifies as follows[1]

$$\nabla \cdot \underline{A} + \frac{1}{c}\frac{\partial V}{\partial t} = -Z_0 j_s(\theta_p, z_p) \quad (22)$$

where $j_s(\theta_p, z_p)$ is the current density flowing "in" or "out" at the location (θ_p, z_p) and Z_0 is the characteristic impendance of the transmission line formed by the plate and the surrounding.

2.1 Boundary Condition at the Termination

For an electric termination at the point (θ_p, z_p)

$$j_s(\theta_p, z_p) = \frac{V_p}{Z_T}\,\delta(\theta - \theta_p)\delta(z - z_p) \quad (23)$$

where V_p is the potential at the point and Z_T the load impedance. Equation (22) with the condition (23) and the expansion of the potentials gives

$$\sum_m \varepsilon_m\left\{\left[\frac{\partial^2 \overline{V}_m}{\partial z^2} + \overline{p}^2\overline{V}_m\right]\cos mg\theta + \left[\frac{\partial^2 \overline{\overline{V}}_m}{\partial z^2} + \overline{\overline{p}}^2\overline{\overline{V}}_m\right]\sin\left(\frac{2m+1}{2}g\theta\right)\right\} =$$

$$= -jk_0\frac{Z_0}{Z_T}V_p\delta(\theta - \theta_p)\delta(z - z_p) \quad (24)$$

The solution of (24) is found to be

$$A^+_{zm} - A^-_{zm} = -\frac{Z_0}{Z_T}V_pQ_m \quad (25)$$

$$V^+_m - V^-_m = 0 \quad (26)$$

where

$$\overline{Q}_m = \frac{2}{\varepsilon_m\varphi_0}\cos mg\theta_p \quad (27)$$

$$\overline{\overline{Q}}_m = \frac{2}{\varepsilon_m\varphi_0}\sin\frac{2m+1}{2}g\theta_p \quad (28)$$

To observe that V_p in eq. (24) is the total voltage at the location of the termination, given as the sum of all the harmonics m and thus still an unknown.

2.2 Boundary Conditions at the Ends

We take into account the current induced by the beam letting in equation (22)

$$j_s = bJ_s\delta(z - z_0) \quad (29)$$

where J_s is the surface current induced by the beam at the ends $z_0 = z_{1,2}$. In the frequency domain, taking into account eqs. (14-16)

$$\sum_m \varepsilon_m\left\{\left[\frac{\partial^2 \overline{V}_m}{\partial z^2} + \overline{p}^2\overline{V}_m\right]\cos mg\theta + \left[\frac{\partial^2 \overline{\overline{V}}_m}{\partial z^2} + \overline{\overline{p}}^2\overline{\overline{V}}_m\right]\sin\frac{2m+1}{2}g\theta\right\} =$$

$$= -jk_0Z_0b\delta(z - z_0)\left[\sum_p \varepsilon_p\tilde{\sigma}_p\cos p(\theta - \theta_0)\right]e^{jkz} \quad (30)$$

Integration of both sides of (30) in the interval $z_0 \pm \varepsilon$ when $\varepsilon \to 0$ gives for the first end at $z = -\ell/2 = z_1$.

$$A_{zm}(z_1) = -bZ_0P_m e^{-jk\ell/2} \quad (31)$$

and for the second end at $z = +\ell/2 = z_2$

$$A_{zm}(z_2) = bZ_0P_m e^{-jk\ell/2} \quad (32)$$

$$P_m = \sum_p \tilde{\sigma}_p h_{pm} \quad (33)$$

with

$$\overline{h}_{pm} = \frac{\cos p\theta_0}{\varepsilon_m}\left\{\text{sinc}(p - mg)\frac{\varphi_0}{2} + \text{sinc}(p+mg)\frac{\varphi_0}{2}\right\} \quad (34)$$

$$\overline{\overline{h}}_{pm} = \frac{\sin p\theta_0}{\varepsilon_m}\left\{\text{sinc}\left(p - \frac{2m+1}{2}g\right)\frac{\varphi_0}{2} - \text{sinc}\left(p + \frac{2m+1}{2}g\right)\frac{\varphi_0}{2}\right\} \quad (35)$$

and sinc(x) = sin(x)/x, assuming that z = 0 is at the center of the plate and that ℓ is the length.

2.3 Determination of the Potential at the Termination

Equations (25,26,31,32) written for the even and odd modes and for the two sides of the plate separated by the termination at $z = z_p$, give a system of eight equations in eight unknown quantities. The solution of the system gives in particular

$$\Delta a^+_m = 2\frac{Z_0}{Z_T}V_pQ_m\frac{k_0}{p}e^{ip\frac{1}{2}}\cos p\left(\frac{\ell}{2} + z_p\right) +$$

$$+ 4bZ_0P_m\frac{k_0}{p}\cos(p - k)\frac{\ell}{2} \quad (36)$$

$$\Delta b^+_m = 2\frac{Z_0}{Z_T}V_pQ_m\frac{k_0}{p}e^{-ip\frac{1}{2}}\cos p\left(\frac{\ell}{2} + z_p\right) +$$

$$+ 4bZ_0P_m\frac{k_0}{p}\cos(p + k)\frac{\ell}{2} \quad (37)$$

$$\Delta = 4i\sin p\ell \quad (38)$$

and, again, several of the symbols can be either $\overline{\ }$ or $\overline{\overline{\ }}$.

Let us consider the case $\theta = 0$ in (13). In this case, only the even modes give contribution to V_p, since

$$V_p = \sum_{m=0}^{\infty}\varepsilon_m\overline{V}_m\,. \quad (39)$$

From (16) and with eqs. (36-38)

$$V_p = \sum_{m=0}^{\infty}\varepsilon_m\left(\overline{a}^+_m e^{-i\overline{p}z_p} + \overline{b}^+_m e^{i\overline{p}z_p}\right) \quad (40)$$

$$V_p = -i\frac{Z_0}{Z_T}V_p\sum_{m=0}^{\infty}\varepsilon_m\overline{Q}_m\frac{k_0}{\overline{p}}\frac{\cos\overline{p}\left(\frac{\ell}{2} + z_p\right)\cos\overline{p}\left(\frac{\ell}{2} - z_p\right)}{\sin\overline{p}\ell} +$$

$$-ibZ_0\sum_{m=0}^{\infty}\varepsilon_m\overline{P}_m\frac{k_0}{\overline{p}}\frac{\left[\cos(\overline{p} - k)\frac{\ell}{2} + \cos(\overline{p} + k)\frac{\ell}{2}\right]}{\sin\overline{p}\ell} \quad (41)$$

which can be solved for V_p to give, for the special case where also $z_p = 0$, that is the termination is at the center of the plate,

$$V_p = \frac{-ibZ_0 \sum_{m=0}^{\infty} \varepsilon_m \bar{P}_m \frac{k_0}{\bar{p}} \frac{\cos k\ell/2}{\sin \bar{p}\ell/2}}{1 + \frac{i}{2}\frac{Z_0}{Z_T} \sum_{m=0}^{\infty} \varepsilon_m \bar{Q}_m \frac{k_0}{\bar{p}} \frac{\cos \bar{p}\ell/2}{\sin \bar{p}\ell/2}} \qquad (42)$$

This can also be written as

$$V_p = \tilde{I}(\omega)\tilde{Z}(\omega) \qquad (43)$$

where

$$\tilde{I}(\omega) = Ne\tilde{f}(k) \qquad (44)$$

is the beam induced current at the angular frequency ω, and

$$\tilde{Z}(\mathbf{w}) = \frac{Z_T Z_p}{Z_T + Z_p} \frac{G(r_0,\theta_0)}{D(\omega)} \qquad (45)$$

is the effective plate impedance. The form factors

$$G(r_0,\theta_0) = 2\frac{\varphi_0}{\pi} \sum_{m=0}^{\infty} \frac{k_0}{\bar{p}} \frac{\cos k\ell/2}{\sin \bar{p}\ell/2} F_m \qquad (46)$$

$$F_m = \frac{1}{2} \sum_s \varepsilon_m \bar{h}_{sm} \frac{I_s(qr_0)}{I_s(qb)} \qquad (47)$$

$$D(\omega) = \sum_{m=0}^{\infty} \frac{k_0}{\bar{p}} \frac{\cos \bar{p}\ell/2}{\sin \bar{p}\ell/2} \qquad (48)$$

show the dependence on the beam position relative to the plate and on the geometry of the plate.

The effective impendance expressed in the form of eq. (45) shows that it can be expressed as the parallel of two impedances, one being the termination itself Z_T and the other given by

$$Z_p = i\frac{Z_0}{\varphi_0} D(\omega) \qquad (49)$$

With a similar method it is straightforth, though quite cumbersome, to calculate the potential V_p also for the case $\theta_p \neq 0$. In this case, also the odd modes will give contribution, but eqs. (43-45) and eq. (48) remain valid.

3. Discussion and Numerical Results

A beam position monitor is made of two parallel plates. Typically the difference of the termination voltages is taken, which is then divided by the sum in order to obtain the beam position.

Inspection of (33) and (34) combined to the form factors (70-72) shows that there is clearly a cutoff in the plate response function given by $\lambda = b\varphi_0$. For the case of long wavelengths, only the mode $m = 0$ gives a significant contribution.

For the special case $\theta_0 = 0$ for one plate, that is $\theta_0 = \pi$ for the other

$$V_\Sigma = 4\frac{Ne}{C}\left[1 + \frac{1}{\varphi_0} \operatorname{arctg} \frac{\left(\frac{r_0}{b}\right)^2 \sin\varphi_0}{1 - \left(\frac{r_0}{b}\right)^2 \cos\varphi_0}\right]\frac{\pi}{2} \qquad (50)$$

$$V_\Delta = 4\frac{Ne}{C}\frac{1}{\varphi_0} \operatorname{arctg} \frac{2\frac{r_0}{b}\sin\frac{\varphi_0}{2}}{1 - \left(\frac{r_0}{b}\right)^2}\frac{\pi}{2} \qquad (51)$$

which are valid for any value of r_0/b between zero and 1. Figures 2 and 3 show the behavior of V_Δ and ratio V_Δ/V_Σ according to eqs. (50 and 51) versus r_0/b for different widths φ_0 of the plates.

References

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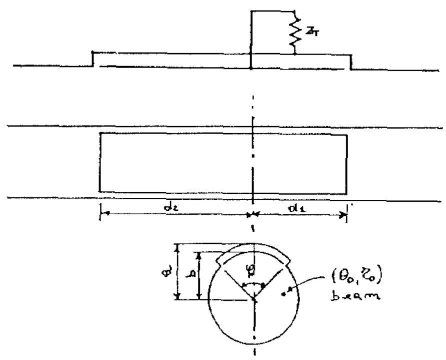

Fig. 1 Geometry of the problem.

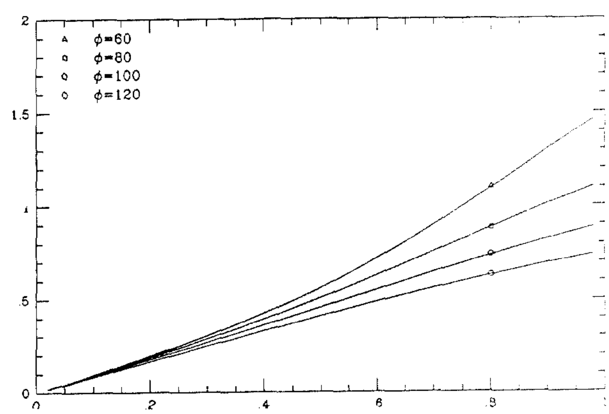

Fig. 2 The difference signal V_Δ divided by $V_0 = 4Ne/C$ versus beam displacement r_0/b, for $\theta_0 = 0$, and for different plate width φ_0.

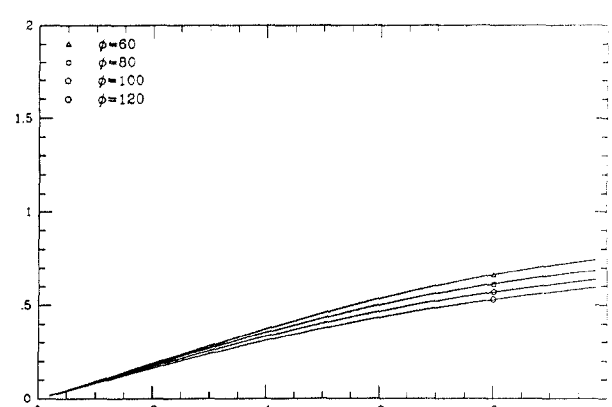

Fig. 3 The ratio V_Δ/V_Σ versus beam displacement r_0/b, for $\theta_0 = 0$, and for different plate width φ_0.