

ELECTRON ENERGY SPECTRUM AND MAXIMUM DISRUPTION ANGLE UNDER MULTI-PHOTON BEAMSTRAHLUNG *

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ABSTRACT

The final electron energy spectrum under multi-photon beamstrahlung process is derived analytically in the classical and the intermediate regimes. The maximum disruption angle from the low energy tail of the spectrum is also estimated. The results are then applied to the TLC and the CLIC parameters.

INTRODUCTION

The synchrotron radiation, called beamstrahlung, due to the beam-beam force is one of the major limitations of the performance for the next generation linear colliders. In addition to the average energy loss, the electron energy spectrum also provides crucial informations. The knowledge on the *tip* of the spectrum reveals the energy resolution for high energy experiments. The spectrum *tail*, on the other hand, gives the probability of low energy particles that will be severely deflected by the same beam-beam field, and would impose constraints on the aperture of the final focusing quadrupole. The concern is relevant because for the next generation linear colliders at the TeV range, the critical energy of radiation is comparable to the beam energy. In such case some particles can in principle lose a large fraction of their initial energies and be deflected by angles much larger than the typical value.

The aim of the present paper is to derive a simple formula for the energy spectrum after successive multi-photon radiations. Similar effort has been done recently by Blankenbecler and Drell¹⁾ Our goal in this paper, however, is to look for compact expressions handy for quick estimations, with attention to the tip and tail of the spectrum. In our derivations we bare in mind that a relatively accurate formula near the initial energy tip is needed for the energy resolution purpose, whereas for background considerations a crude estimation near the low energy tail is enough.

THE RATE EQUATION

Let $\psi(E, t)$ be the energy spectral function of electrons at time t normalized as $\int \psi(E, t) dE = 1$. We assume that the emission of a photon takes place in an infinitesimally short time interval. Then the evolution of the spectral function can be described by the rate equation

$$\frac{\partial \psi}{\partial t} = -\nu(E)\psi(E, t) + \int_E^\infty F(E, E')\psi(E', t)dE' \quad (1)$$

where the first term corresponds to the *sink*, and the second term the *source*, for the evolution of $\psi(E, t)$. Here, $\nu(E)$ is the average number of photons radiated per unit time and F is the spectral

function of radiation; i.e., $F(E, E')dE'$ is the transition probability of an electron from energy E' to the energy interval $(E, E+dE)$ per unit time. Obviously, $F(E, E') = 0$ if $E \geq E'$. Notice, however, that F does not include the probability for electrons to remain at the same energy without photon emission.

An important parameter characterizing the spectral function is the critical energy ω_c of radiation. Normalizing it by the energy E before emission, we define

$$\xi(E) = \frac{\omega_c}{E} = \frac{3\lambda_c\gamma^2}{2\rho} \equiv \frac{E}{K} \quad (2)$$

where γ is the Lorentz factor for energy E , λ_c the Compton wave length, and ρ the radius of curvature. Since $\rho \propto \gamma$, the introduced parameter K is independent of energy.

The quantum-theoretical spectral function F was derived by Sokolov and Ternov:²⁾

$$F(E', E) = \frac{K\nu_{cl}}{EE'} f(\xi, y) \quad (3)$$

$$f(\xi, y) = \frac{3}{5\pi} \frac{1}{1+\xi y} \left[\int_y^\infty K_{5/3}(x) dx + \frac{\xi^2 y^2}{1+\xi y} K_{2/3}(y) \right],$$

where $y = K[(1/E') - (1/E)]$, K_ν 's are the modified Bessel functions and ν_{cl} is the number of photons per unit time calculated by the classical formula,

$$\nu_{cl} = \nu_{E=0} = \frac{5}{2\sqrt{3}} \frac{\alpha\gamma}{\rho} \quad (4)$$

α being the fine structure constant. Note that for a given field strength ν_{cl} is independent of the particle energy.

By definition, $\nu(E)$ is given by integrating F as

$$\nu(E) = \int_0^E F(E', E) dE' \equiv \nu_{cl} U_0(\xi) \quad (5)$$

The function $U_0(\xi)$ is normalized such that $U_0(0) = 1$, and can be represented by the following approximate expression:

$$U_0(\xi) = \frac{1 - 0.598\xi + 1.061\xi^{5/3}}{1 + 0.922\xi^2} \quad (6)$$

where the relative error is less than 0.7% for any ξ .

CLASSICAL REGIME

Let us solve Eq.(1) for constant fields ($K = \text{const}$) in the classical regime, i.e., $\xi_0 \ll 1$. Instead of using the exact Sokolov-Ternov formula for $f(\xi, y)$ as in Eq.(3), we invoke an approximate spectral function

$$g(y) = \frac{1}{\Gamma(1/3)} y^{-2/3} e^{-y} \quad (7)$$

which gives a reasonable approximation for any ξ and y . The advantages of using $g(y)$ is that it is a function of y only and it provides a simple Laplace transform.

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The rate equation can now be solved exactly with $\nu(E) = \nu_{cl}$ in the classical limit. Changing variables to $\eta \equiv K/E$ and $\varphi(\eta, t) \equiv E\psi(E, t)$, the rate equation can be rewritten as

$$\frac{\partial \varphi}{\partial t} = -\nu_{cl}\varphi + \nu_{cl} \int_0^\eta g(\eta - \eta')\varphi(\eta', t)d\eta' \quad (8)$$

Applying the following Laplace transformations,

$$\begin{aligned} \hat{\varphi}(p, t) &= \int_0^\infty e^{-\eta p} \varphi(\eta, t) d\eta \quad ; \\ \hat{g}(p) &= \int_0^\infty e^{-\eta p} g(\eta) d\eta = \frac{1}{(1+p)^{1/3}} \quad ; \end{aligned} \quad (9)$$

Eq.(8) takes the form

$$\frac{\partial \hat{\varphi}}{\partial t} = \nu_{cl} \left[-1 + \frac{1}{(1+p)^{1/3}} \right] \hat{\varphi} \quad , \quad (10)$$

which can be solved immediately to obtain

$$\hat{\varphi}(p, t) = \hat{\varphi}(p, 0) \exp[N_{cl}(-1 + (1+p)^{-1/3})] \quad , \quad (11)$$

with $N_{cl} = \nu_{cl}t$ being the average number of photons radiated up to the time t .

Let us assume the initial condition $\psi(E, 0) = \delta(E - E_0)$. Then from the inverse Laplace transformation we get

$$\psi(E, t) = \frac{e^{-N_{cl}y}}{\xi_0 E} \frac{1}{2\pi i} \int_{\lambda-i\infty}^{\lambda+i\infty} \exp(N_{cl}p^{-1/3} + yp) dp \quad , \quad (\lambda > 0) \quad (12)$$

with $y = \eta - \eta_0$ and $\eta_0 = K/E_0$. Using p instead of yp , we get

$$\psi(E, t) = e^{-N_{cl}} \left[\delta(E - E_0) + \frac{e^{-y}}{E_0 - E} h(y^{1/3} N_{cl}) \right] \quad , \quad (13)$$

with

$$h(x) = \frac{1}{2\pi i} \int_{\lambda-i\infty}^{\lambda+i\infty} \exp(xp^{-1/3} + p) dp = \sum_{n=1}^{\infty} \frac{x^n}{n! \Gamma(n/3)} \quad , \quad (14)$$

where $\lambda > 0$ and $0 \leq x \leq \infty$.

The first term of Eq.(13) represents the electron population that suffers no radiation. In addition, each term of the Taylor expansion in Eq.(14) has a physical meaning: The n^{th} term corresponds to n times of iterations on the radiation spectral function F , thus represents the process of n -photon emissions. For a given x , the largest contribution comes from the term $n \sim \sqrt{3x} = \sqrt{3N_{cl}y}^{1/6}$. Therefore for a finite N_{cl} the tail of the spectrum is not dominated by single photon emissions, but by multi-photon emissions.

Applying the saddle point method to Eq.(14), we can find the asymptotic form for $h(x)$ at $x \gg 1$. Based upon this asymptotic form an approximate expression is available:

$$h(x) \sim \sqrt{\frac{3}{8\pi}} \left[\frac{\sqrt{x/3}}{1 + 0.53x^{-5/6}} \right]^{3/4} \exp[4(x/3)^{3/4}] \quad , \quad (15)$$

which is accurate within 2% for any $0 \leq x \leq \infty$.

INTERMEDIATE REGIME

For finite values of ξ , the rate equation cannot be solved exactly since $\nu(E)$ is not constant any more. However, in the intermediate regime where $\xi \lesssim \mathcal{O}(10)$, $\nu(E)$ should not deviate from ν_{cl} too significantly. This suggests to a solution based upon minor perturbations to the classical result of Eq.(13).

The first, and natural, attempt is to replace N_{cl} in the exponent by the photon number calculated by the quantum theory:

$$N_\gamma = \nu(E_0)t = U_0(\xi_0)N_{cl} \quad . \quad (16)$$

This replacement is good, however, only near the high energy tip $E \sim E_0$. It turns out that a better approximation exists if, in addition, one replaces the second N_{cl} in the argument of $h(x)$ by the following empirical formula:

$$\bar{N}(E) = \frac{1}{1 + \xi_0 y} N_{cl} + \frac{\xi_0 y}{1 + \xi_0 y} N_\gamma \quad , \quad (17)$$

which reduces to N_{cl} at $E = E_0$ (or, $y = 0$). Thus we have

$$\psi(E, t) = e^{-N_\gamma} \left[\delta(E - E_0) + \frac{e^{-y}}{E_0 - E} h(y^{1/3} \bar{N}(y)) \right] \quad . \quad (18)$$

An alternative approach is the following: One may retain the second N_{cl} while replacing the first one with an effective photon number. Since the particle with final energy E must have been cascading down through all energies between E_0 and E statistically, an effective photon number can be given by

$$\hat{N}(E) = \frac{1}{2}(\nu(E_0) + \nu(E))t = \frac{1}{2} \left[U_0(\xi_0) + U_0\left(\frac{\xi_0}{1 + \xi_0 y}\right) \right] N_{cl} \quad (19)$$

In this approach the solution becomes

$$\psi(E, t) = e^{-\hat{N}(E)} \left[\delta(E - E_0) + \frac{e^{-y}}{E_0 - E} h(y^{1/3} N_{cl}) \right] \quad (20)$$

It turns out that the second solution agrees better with the simulation result based on the approximate spectral function, whereas the first fits better with the exact spectral function of Sokolov-Ternov. Without obvious advantage of either one, we shall simply adopt Eq.(18) for the intermediate regime.

MAXIMUM DISRUPTION ANGLE

Particles that suffer severe energy losses would be disrupted with large angles by the strong beam-beam field. A simulation was done by monitoring low energy test particles throughout the collision process. The maximum deflection angle for a given energy εE_0 , where $\varepsilon \ll 1$, is found to be roughly

$$\theta_{max} \sim \frac{\sigma}{\sigma_z} \frac{D/\varepsilon}{\sqrt{1 + (0.75D/\varepsilon)^{4/3}}} \quad , \quad (\varepsilon \ll 1) \quad (21)$$

where $D = D_{x,y}$ is the *disruption parameter*

$$D_{x,y} \equiv \frac{2r_c N \sigma_z}{\gamma \sigma_{x,y} (\sigma_x + \sigma_y)} \quad , \quad (22)$$

for the x and y dimensions, respectively, and $\sigma = \sigma_{x,y}$.

Table 1. Parameters for TLC and CLIC

	TLC	CLIC
$E_0[\text{TeV}]$	0.5	1.0
N	8×10^9	5×10^9
$\sigma_x, \sigma_y[\text{nm}]$	190, 1	60, 12
$\sigma_z[\mu\text{m}]$	26	200
$\epsilon_x, \epsilon_y [10^{-12}\text{mrad}]$	2.58, 0.023	1.53, 0.51
D_x, D_y	0.033, 6.27	0.667, 3.333
ξ_1	3.43	1.48
* δ	0.15	0.25
* N_γ	1.33	3.0
ϵ_{min}	0.013	0.03
$\theta_{x,max}, \theta_{y,max}[\text{mrad}]$	10, 0.4	1.0, 0.4

* Quantities computed by simulations.

The minimum value of ϵ can in principle be as small as $1/\gamma$. But the real problem is about how small a ϵ should one care. Since the number of photons N_γ per beam particle for linear colliders in the near future is of order unity, the spectral function $\psi(\epsilon E_0)$ given in Eq.(18) is always dominated by the factor ϵ^{-y} in the spectrum tail, where $y \gg 1$ (in logarithmic sense). Therefore if the acceptable background counts is n out of N electrons, the minimum ϵ of concern is approximately determined by $y = \log(N/n)$, or

$$\epsilon_{min} = \frac{1}{1 + \xi_1 \log(N/n)} \quad (23)$$

With this value of ϵ , one can directly estimate the maximum deflection angle using Eq.(21). Since the dependence on n is only logarithmic, one can set $n = 1$ for practical purposes.

APPLICATION TO TEV COLLIDERS

We now apply the formula derived above to the specific TeV collider design. So far we have considered constant fields only, but in reality the beam-beam field varies in time like a Gaussian function. Thus K (and, therefore, ξ_0) is not a constant in time. The parameter ξ_0 has to be replaced with a typical value of ξ during the collision. We suggest to use

$$\xi_1 \equiv \frac{r_e^2 \gamma N}{\alpha \sigma_z} \frac{2}{\sigma_x + \sigma_y}, \quad (24)$$

where r_e is the classical electron radius, σ_x , σ_y and σ_z are the horizontal, vertical, and longitudinal *r.m.s.* beam sizes at the collision point, respectively. This expression is larger than the average ξ by a factor about 3/2, but it provides better agreements with simulations. The reason is that the low energy tail of the spectrum is dominated by the radiation with larger local ξ , and is therefore weighted more; whereas the high energy tip of the spectrum is relatively insensitive to the choice of ξ .

In addition, the N_{cl} at the end of the entire collision process has been derived to be³⁾

$$N_{cl} = 1.06 \frac{2\alpha r_e N}{\sigma_x + \sigma_y}. \quad (25)$$

Computer simulations for Gaussian beams have been performed using the program code ABEL⁴⁾ on the linear collider design parameters for TLC,⁵⁾ and that for CLIC.⁶⁾ The adopted parameters are summarized in Table.1. The disruption effect is included in the simulation.

The analytic formula Eq.(18) (dots) is compared with the simulations (histogram) in Fig. 1. The agreement is excellent for the TLC parameter, while there is a slight discrepancy at the low-energy tail for the CLIC parameter. The reason is that the field is enhanced somewhat in the CLIC case due to the more significant pinch effect in the horizontal plane. Since in general the spectrum tail is sensitive to various parameters, especially ξ , the excellent agreement seen in the figure is somewhat fortuitous.

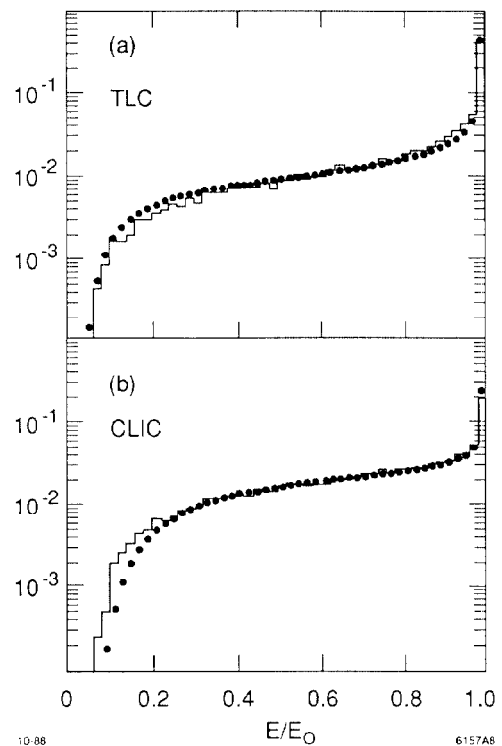


Fig. 1. Electron energy spectrum for TLC and CLIC.

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