LONGITUDINAL INSTABILITY OF AZIMUTHALLY-NONSYMMETRIC BEAM IN A SYNCHROTRON

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Introduction and Summary

Usually a theoretical study of longitudinal instabilities implies the concept of a closed beam made of identic and equidistantly spaced bunches. As a matter of fact, such an azimuthal symmetry of the beam can be violated. It occur either because of bunch-to-bunch spread of parameters (i) or because of partial orbit filling by the beam (ii). The latter is peculiar to large proton synchrotrons of the UNK-type due to the injection schemes accepted. The paper presents an approach to the problem of longitudinal instabilities in this case. The features peculiar to beam oscillation eigenmodes are discussed. The results of computations for the UNK are enclosed.

Major Set of Equations

The beam-excited electric field can be written as $\sum_k E_k(\Omega)e^{ik\vartheta-i\Omega t}$, where t is the time, Ω is the frequency of coherent oscillations, ϑ is the azimuth in co-rotating coordinate system (namely, $\vartheta = \theta-\omega_s t$, θ is the generalized azimuth around the ring in the laboratory system, ω_s is the angular velocity of a synchronous particle). The amplitude of the electric field harmonic $E_k(\Omega)$ is directly proportional to the beam current harmonic $J_k(\Omega)$:

$$E_k(\Omega)= - Z_k(\Omega)J_k(\Omega) \;/\; L, \qquad (1)$$

where L is the orbit length, $Z_k(\Omega)$ is the longitudinal impedance of the vacuum chamber. The amplitude of beam current harmonic can be presented as a sum over the contributions from all the bunches:

$$J_k = \sum_j J_k^{(j)} \exp(-ik\vartheta_j), \qquad (2)$$

where ϑ_j is the coordinate of the j-th bunch center, $J_k^{(j)}$ is the current harmonic of the j-th bunch calculated in the coordinate system attached to its center; the summation is performed over all the beam bunches.

The field (1) causes the onset of time-depedent perturbations on the background of the initial steady-state inside-the-bunch particle distributions. Following papers [1,2], one obtains an infinite set of equations in terms of bunch current perturbation harmonics $J_k^{(j)}$:

$$J_k^{(j)}= \sum_{k'=-\infty}^{\infty} Y_{kk'}^{(j)}(\Omega)(Z_{k'}(\Omega)/k') \;\times \qquad (3)$$

$$\times \sum_{j'} \exp\left[-ik'(\vartheta_{j'}-\vartheta_j)\right] J_{k'}^{(j')}, \quad k' \neq 0.$$

The expressions for the so called dispersion integrals $Y_{kk'}^{(j)}$ can be found elsewhere [1,2].

It seems evident, that the system (3) can be rewritten in terms of the beam harmonics J_k [3]. But this form is less convenient in our case. The reason lies in the fact, that both $J_k^{(j)}$ and $Y_{kk'}^{(j)}$ can be almost independent of their subscripts, what we are going to take the advantage of below.

The problem (3) can be hardly solved in its general form. Therefore let us introduce some simplifications:

a) The dispersion integral $Y_{kk'}^{(j)}$ can be expanded in series in the multipole excitations. The individual multipoles are excited independently, provided the impedance is a sufficiently low-frequncy one. The latter allows the use of the so called approximation of the uncoupled multipoles:

$$Y_{kk'}^{(j)}= \sum_{m'=-\infty}^{\infty} Y_{kk'}^{(j,m')} \simeq Y_{kk'}^{(j,m)}, \qquad (4)$$

where m is the usual multipole index. From now on we take $m>0$ imposing no restrictions on the generality.

Whithin the frames of the above approximation the quantities $Y_{kk'}^{(j)}$ acquire the following properties:

$$Y_{-k,-k'}^{(j)}\simeq Y_{kk'}^{(j)}, \quad Y_{-kk'}^{(j)}\simeq Y_{k,-k'}^{(j)}\simeq(-1)^m Y_{kk'}^{(j)}. \qquad (5a)$$

Hereupon one arrives at a simple coupling condition for the curruent harmonics:

$$J_{-k}^{(j)}\simeq (-1)^m J_k^{(j)}. \qquad (5b)$$

(The approximate equalities (5) become the exact ones provided the synchroton oscillations are taking place in a symmetric potential well.)

b) Let the impedance $Z_k(\Omega)$ be sufficiently nontrivial only for the harmonics numbered within the range:

$$k \simeq \pm \;(k_0 \pm \Delta k). \qquad (6)$$

In this case one should take into account only that harmonics of current which are numbered by the subscripts of the same range. Simultaneously, Eqs.(5) allow the reduction of the problem to the positive values of k,k'. We also suppose that

$$k_0 \;\Delta\vartheta_b \gtrsim 1, \quad \Delta k \;\Delta\vartheta_b << 1, \qquad (7)$$

where $\Delta\vartheta_b$ is the typical length of a bunch along the azimuth. It is this condition that results in a weak dependence of the quantities $Y_{kk'}^{(j)}$ of their subscripts. One can see from Eq.(3) that the values of bunch current harmonics are also kept almost unchanged, provided their index ranges in bounds (6). Therefore it is sufficient to retain the only equation, e.g. for the harmonic numbered by $k_0>0$:

$$J^{(j)}= \sum_{j'} a_{jj'}(\Omega) \; J^{(j')}, \qquad (8a)$$

$$a_{jj'}(\Omega)= Y^{(j)}(\Omega) \; G(\vartheta_{j'}-\vartheta_j \;,\Omega), \qquad (8b)$$

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$$G(\vartheta,\Omega)=\sum_{k=-\infty}^{\infty}\frac{Z_k(\Omega)}{k}\exp(-ik\vartheta),\ k\neq 0, \quad (8c)$$

where to simplify the notataions we substitute $J^{(j)}_{k_0} \rightarrow J^{(j)}$, $Y^{(j)}_{k_0 k_0} \rightarrow Y^{(j)}$.

Identic Bunches with Different Population

The solution of the problem involved becomes easier if the bunches are supposed to differ by their population only. In this case all the dispersion integrals take the form

$$Y^{(j)}(\Omega) = \nu_j Y(\Omega), \quad (9)$$

where ν_j is the relative population of the j-th bunch; by definition $\max(\nu_j) = 1$. The dispersion equation of instability can be written as

$$1 = Y(\Omega)\ \mu_n(\Omega), \quad (10)$$

where μ_n is the n-th eigenvalue of the problem:

$$\mu J^{(j)} = \sum_{j'} \nu_j G(\vartheta_{j'} - \vartheta_j, \Omega) J^{(j')}. \quad (11)$$

The corresponding eigenvectors define the set of linearly-independent (but not always mutually orthogonal) beam oscillation modes. It is evident that the total number of these modes coincides with the amount of the bunches located around the orbit.

Eq.(10) can be studied by the well-known threshold plot technique [2]. The quantity μ_n is interpreted as the effective impedance of the n-th mode. The whole system is stable when all the points $1/\mu_n$ are located either beyond the region encircled by the threshold curve Y(ReΩ+i0) on the comlex plane Y, or directly on the ReY = 0 axis. The stability criterion can be easily reformulated in terms of the impedance plane $Z \sim Y^{-1}$, which we employ in what follows.

Let us study some general features of the solutions of the problem (11).

Usually one gets $|\Omega| \ll \Delta k\omega_s$. Therefore $G(\vartheta,\Omega) \simeq G(\vartheta,0)$ and eigenvalues μ (as well as eigenvectors $\vec{J}$) are no more frequency Ω dependent. From the physical point of view the latter means that Robinson-type instabilities [4] become insufficient. Now $G = -G^*$ and the quantities

$$(\mu, \vec{J}) \text{ and } (\mu' = -\mu^*, \ \vec{J}' = \vec{J}^*)$$

will present a pair of eigenvalues and eigenvectors of the problem involved. Therefore:

a) eigenvalues μ are symmetric with respect to the imaginary axis. Each pair of symmetric μ, $\mathrm{Re}\mu \neq 0$ corresponds to the mutually complement (otherwise, complex conjugated) beam oscillation modes. Any of these pair modes can become unstable either before ($\mathrm{Re}\mu < 0$) or after ($\mathrm{Re}\mu > 0$) gamma-transition. (One should take into account the one-sided shape of the threshold plots under assumption (4).)

b) Purely imaginary μ's stand for the (almost)stable (up to the Robinson effect [4]) modes. The latter are always characterized by either in-phase or counter-phase oscillations of any two adjacent bunches: for $\mu = -\mu^*$ $\vec{J} = \vec{J}^*$. There is at least one such mode in a beam consisting of an odd number of bunches.

Let us consider one important and well-known case - the azimuthally-symmetric beam:

$$\nu_j = 1, \quad \vartheta_j = -(2\pi/M)\ j\ , \quad (12)$$

where M is the total number of bunches. Now problem (11) can be solved in analytics. Its eigenvectors constitute an orthogonal and normalized to unity coordinate basis:

$$J^{(j)}_n = M^{-1/2}\exp(-2\pi i n j/M), \quad (13)$$

$$\langle \vec{J}_n | \vec{J}_{n'} \rangle = \delta_{nn'} = \begin{cases} 0, n\neq n' \\ 1, n=n' \end{cases}$$

and define M normal modes of the coupled bunch oscillations. They can be discriminated by the value of synchrotron frequency phase shift between two adjacent bunches, $\Delta\Psi = 2\pi n/M$. The eigenvalue of the n-th mode can be found as

$$\mu^{(0)}_n = M\sum_{l=-\infty}^{\infty} Z_{n+lM}(\Omega)/(n+lM),\ n+lM\neq 0. \quad (14)$$

This particular solution of the problem at issue can help us arrive at an important conclusion about the location of μ_n values on the complex plane. Indeed, let us suppose that the initially symmetric beam partially looses population of any of its bunches in an arbitrary way, some of the bunches being knocked away altogether. Nevertheless, the newly-formed beam eigenvectors can be expanded in series in basis (13) vector components, while its eigenvalues $\mu_{n'}$ can be presented as quadric bilinear functional induced by matrix (11) of the initial beam. Making sufficient use of orthogonality condition (13), one readily arrives at

$$\mu_{n'} = \zeta_{n'} \sum_n \xi_{n'n}\mu^{(0)}_n, \quad \sum_n \xi_{n'n} = 1, \quad (15)$$

where the summation is performed over all the normal modes (13),(14); $0 \leq \zeta_{n'} \leq 1$, $0 \leq \xi_{n'n} \leq 1$.

Eq.(15) has the following geometrical interpretation: any of $\mu_{n'}$ eigenvalues lies inside the boundaries of the minimal (least-area) convex poligon circumscribed around all the points $\mu^{(0)}_n$ of the complex plane μ. When the initial beam (12) is stable, all the points $\mu^{(0)}_n$ are located inside the stability region of the threshold plot on Z-plane. Therefore all the points $\mu_{n'}$, representing the derivative beam, would also belong to the stable area by virtue of (15). Thus, one arrives at the general conclusion: the beam ,which is made nonsymmetric either by partial loss of some of its bunch population or(and) by total knock-out of arbitrary bunches, will never be more unstable than the initial symmetric beam.

Instability Threshold Computations

To proceed to computer calculations let us take the narrow-band cavity impedance

$$Z_k(\Omega)= R_{sh}\times\left[1-i\,\frac{(k\omega_s+\Omega)^2-\omega_0^2}{2(k\omega_s+\Omega)\,\Delta\omega}\right]^{-1}, \qquad (16)$$

where R_{sh} is the coupling (shunt) impedance, ω_0 is the resonant frequency, $\Delta\omega$ is the cavity bandwidth. In this case sum (8в) can be easily summed up to yield

$$G(\vartheta,\Omega)\Big|_{0<\vartheta\leq 2\pi} \simeq \frac{4\pi R_{sh}\Delta\omega}{\omega_1-\omega_2}\times \qquad (17)$$

$$\times\left\{\frac{\omega_1}{\omega_1-\Omega}\left[\frac{1}{-2\pi i k_1}+\frac{\exp(-ik_1\vartheta)}{1-\exp(-2\pi i k_1)}\right]-(1\rightarrow 2)\right\},$$

$$G(\vartheta\pm 2\pi,\Omega)=G(\vartheta,\Omega),\quad \omega_{1,2}=\pm\sqrt{\omega_0^2-\Delta\omega^2}-i\Delta\omega,$$

$$k_{1,2}=(\omega_{1,2}-\Omega)/\omega_s.$$

Eigenvalues $\mu_n^{(0)}$ for the azimuthally-symmetric beam take the form:

$$\mu_n^{(0)} \simeq \frac{4\pi R_{sh}\Delta\omega}{\omega_1-\omega_2}\times \qquad (18)$$

$$\times\left[\frac{\omega_1}{\omega_1-\Omega}\left[\frac{\delta_{n0}}{-2\pi i(k_1-n)/M}+\frac{\exp(-2\pi i(k_1-n)/M)}{1-\exp(-2\pi i(k_1-n)/M)}\right]-(1\rightarrow 2)\right].$$

The UNK beam will consist of the sequence of batches bunched by RF frequency $q\omega_s$ (q - harmonic number) and separated by time intervals. The number of bunches at the injection flat-top of the 1st stage will be as great as $M \simeq (1\div 12)\cdot 10^3 >> 1$. Therefore the direct computer solution of the problem (11) is hardly possible because of its great dimension. But this complication can be easily overcome. Indeed, if one finds an integer $K > 0$ for which

$$K|\omega_1 - lq\omega_s|/q\omega_s << 1,\ l \text{ is an integer}, \qquad (19)$$

then $J^{(j+K)} \simeq J^{(j)}$. Therefore the dimension of problem (11) can be reduced to $N \simeq M/K$, the latter means that each K adjacent bunches are formally treated as a single one. The eigenvalues we are interested in (namely, the largest by the modulus ones) would practically coincide.

Fig.1 presents the results of computations illustating the method convergence. We study the single batch of bunches in the 1st stage of the UNK ($M = 10^3$, $q = 1.4\cdot 10^4$), interacting with the accelerating cavities ($\Delta\omega/\omega_0 = 2\cdot 10^{-4}$, $(\omega_0 - q\omega_s)/\Delta\omega = 0.75$). The parameter plotted is the matrix dimension N.

Fig.2 shows the results of computations of the effective impedances, i.e. eigenvalues μ_n, which correspond to the operation mode of the UNK accelerating cavities. The injection flat-top is studied. The beam and cavities parameters are taken to be: E=70 GeV, J_0=1.4 A, R_{sh}=16x0.25=4 MOhm, $\Delta p/p_s = \pm 2.1\cdot 10^{-3}$, $q\Delta\vartheta_b/2\pi = 0.54$. Figures 1÷12 denote the number of consequent batches on the orbit, their minimal separation being 100 empty buckets. Curve 13 is the plot of eigenvalues $\mu_n^{(0)}$ for the symmetric beam filling the whole orbit. One can easily find out that the nodes of the brocken lines 1÷12 и 13 are in fact satisfying condition (15). The threshold curves for the dipole and quadrupole oscillations (m=1,2) are plotted by the continuous curves. The stable area lies near the coordinate origin and in the left half-plane. It can be seen, that all the bunch-to-bunch coupled modes of dipole oscillations of 1 and 2 consequent batches are located below the instability threshold. The quadrupole oscillations will become unstable provided 9 or more batches are circulating on the orbit. The higher modes of multipole oscillations ($m \geq 3$) are stable. To provide the beam longitudinal stability a special feedback system is being developed.

unk stage#1.injection flat-top:1batch

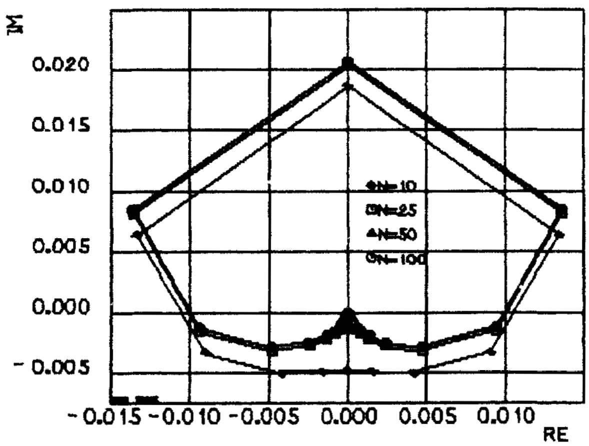

fig1. convergence test

unk stage#1.injection flat-top:1-12batches

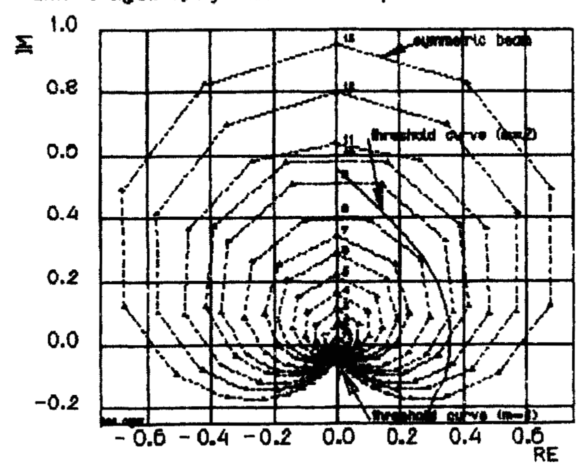

fig2. eigenvalues, relative units

<u>Referencies</u>

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