Amplitude growth due to random, correlated kicks

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1 Introduction.

Historically, stochastic processes, such as gas scattering or stochastic cooling, have been treated by the Fokker-Planck equation.[1] In this approach, usually considered for one dimension only, the equation can be considered as a continuity equation for a variable which would be a constant of the motion in the absence of the stochastic process, for example, the action variable, $I = \epsilon/2\pi$ for betatron oscillations, where ϵ is the area of the Courant-Snyder ellipse, or energy in the case of unbunched beams, or the action variable for phase oscillations in case the beam is bunched. A flux, Φ, including diffusive terms can be defined, usually to second order.

$$\Phi = M_1 F(I) + M_2 \partial F/\partial I + \cdots$$

M_1 and M_2 are the expectation values of of δI and $(\delta I)^2$ due td the individual stochastic kicks over some period of time, T, long enough that the variance of these quantities is sufficiently small. Then the Fokker-Planck equation is just

$$\partial F/\partial I + \partial\Phi/\partial I = 0 \quad .$$

In many cases, those where the beam distribution has already achieved its final shape, usually Gaussian, it is sufficient to find the rate of increase of $\langle I \rangle$ by taking simple averages over the Fokker-Planck equation.

At the time this work was begun, there was good knowledge of the second moment for general stochastic processes due to stochastic cooling theory, but the form of the first moment was known only for extremely wideband (short correlation times) processes, such as gas scattering.[4] The purposes of this note are to derive an expression relating the expected single particle amplitude growth to the noise autocorrelation function and to obtain, thereby, the form of M_1 for narrow band processes (long correlation time).

2 Localized kicks as additive noise.

We shall describe the dynamics in terms of extended phase space coordinates x, p, and θ.

$$\mathbf{x} = \begin{pmatrix} x \\ p \end{pmatrix} \equiv \begin{pmatrix} x \\ \alpha x + \beta x' \end{pmatrix} = \sqrt{2I\beta} \begin{pmatrix} \sin(\tilde{\psi} + \delta) \\ \cos(\tilde{\psi} + \delta) \end{pmatrix} \quad . \tag{1}$$

Here, $\beta(\theta)$, $\alpha(\theta)$, and $\psi(\theta)$ are the usual Courant-Snyder lattice functions which express the Floquet solutions to Hill's equation, and we define $\tilde{\psi}(\theta) = \psi(\theta) - \nu\theta$, where ν is the tune. (Note that $\tilde{\psi}$ is a periodic function of θ.) I and δ are canonically conjugate variables (in the same symplectic form as x and p). A single turn through a perfect machine is represented by a linear mapping,

$$\mathbf{x}(\theta + 2\pi) = \mathbf{R}\mathbf{x}(\theta) \quad ,$$

where $\mathbf{R}$ is the rotation matrix

$$\mathbf{R} = \begin{pmatrix} \cos 2\pi\nu & \sin 2\pi\nu \\ -\sin 2\pi\nu & \cos 2\pi\nu \end{pmatrix}$$

The "emittance" associated with a particle at $\mathbf{x}$ is π times the Courant-Snyder invariant.

$$\begin{aligned} \epsilon &= \pi[x^2 + (\alpha x + \beta x')^2]/\beta \\ &= \frac{\pi}{\beta}\mathbf{x}^T\mathbf{x} \\ &= 2\pi I \end{aligned}$$

We have put "emittance" in quotation marks because one cannot properly speak of the emittance of a single particle. More correctly, this is the phase space area enclosed by Courant-Snyder tori.

Consider first a single particle circulating in a perfect storage ring and receiving small, random, localized kicks at one location in the ring. Such a dynamical system is described by the stochastic process,

$$\mathbf{x}_k = \mathbf{R}\mathbf{x}_{k-1} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} N_{k-1} \tag{2}$$

where $\mathbf{x}_k$ is the state of the particle after k turns, and $\{ N_k \mid k = 0, 1, 2, \ldots \}$ is a set of random variables ("noise"). In one turn, the "emittance" of the particles will change according to

$$\begin{aligned} \epsilon_k &= \frac{\pi}{\beta}\mathbf{x}_k^T\mathbf{x}_k \\ &= \frac{\pi}{\beta}\left[\mathbf{x}_{k-1}^T\mathbf{R}^T\mathbf{R}\mathbf{x}_{k-1} + 2(0\ N_{k-1})\mathbf{R}\mathbf{x}_{k-1} + N_{k-1}^2 \right] \\ &= \epsilon_{k-1} + \frac{2\pi}{\beta}[\cos(2\pi\nu)\, p_{k-1} N_{k-1} - \sin(2\pi\nu)\, x_{k-1} N_{k-1}] \qquad (3) \\ &\quad + \frac{\pi}{\beta}N_{k-1}^2 \quad , \qquad (4) \end{aligned}$$

where I've used $\mathbf{R}^T\mathbf{R} = \mathbf{1}$. This can be written in a different form that uses only polar variables by substituting from Eq.(1).

$$I_k = I_{k-1} - N_{k-1}\sqrt{2I_{k-1}/\beta}\,\sin(2\pi\nu + \tilde{\psi} + \delta_{k-1}) + \frac{1}{2\beta}N_{k-1}^2$$

We now want to average Eq.(4) over all possible noise histories. As an initial calculation, let us suppose that N_k is a zero mean process uncorrelated with the state.

$$\langle p_{k-1} N_{k-1} \rangle = \langle x_{k-1} N_{k-1} \rangle = 0 \quad . \tag{5}$$

Then, the expected "emittance" grows as follows.

$$\langle \epsilon_k \rangle = \langle \epsilon_{k-1} \rangle + \frac{\pi}{\beta}\langle N_{k-1}^2 \rangle \tag{6}$$

If N_k is zero mean, stationary noise associated with random fluctuations in a dipole field, δB_k, then

$$N_k = \beta \frac{Bl}{|B\rho|}\frac{\delta B_k}{B} \quad . \tag{7}$$

Substituting this into Eq.(6) yields,

$$\langle \epsilon_k \rangle = \langle \epsilon_{k-1} \rangle + \pi\beta \left(\frac{Bl}{|B\rho|} \right)^2 \left\langle \left(\frac{\delta B}{B} \right)^2 \right\rangle \quad ,$$

where I've used the stationary hypothesis to eliminate the subscript k on δB. The growth rate of expected "emittance" is

$$\begin{aligned} r = d\langle\epsilon\rangle/dt &= f \times (\langle \epsilon_k \rangle - \langle \epsilon_{k-1} \rangle) \\ &= \pi f \beta \left(\frac{Bl}{|B\rho|} \right)^2 \left\langle \left(\frac{\delta B}{B} \right)^2 \right\rangle \end{aligned}$$

where f is the frequency of rotation through the accelerator. Putting in some units,

$$\begin{aligned} r[10^{-3} \times \pi \ \ \mathrm{mm-mr/hr}] \ \cong \ & f[50\mathrm{kHz}] \times \beta[100\mathrm{m}] \times \\ & \left(\frac{Bl}{|B\rho|}[2\pi/774] \right)^2 \times \left\langle \left(\frac{\delta B}{B}[10^{-7}] \right)^2 \right\rangle \end{aligned}$$

That is, for a revolution frequency of 50 kHz, and $\beta\gamma = 1000$ ($E \approx 1$ TeV), a Tevatron dipole positioned where $\beta = 100$ meters with an rms fluctuation in $\delta B/B$ of 10^{-7} will increase the expected *invariant* "emittance" by π mm-mr per hour.

We now relax the zero correlation assumption of Eq.(5). Let us assume that we start from some fixed state, $\mathbf{x}_0$, which then evolves according to Eq.(2) where N_k is a zero mean, stationary, stochastic process with autocorrelation $\langle N_k N_m \rangle \neq 0$. Since N is stationary, $\langle N_k N_m \rangle$ depends only on the difference $k - m$, and we shall define the singly indexed autocorrelation function,

$$\Phi_{k-m} = \Phi_{m-k} \equiv \langle N_k N_m \rangle \quad .$$

Now, since the noise is assumed to have zero mean, we have

$$\forall k : \quad \langle N_k \mathbf{x}_0 \rangle = \langle N_k \rangle \mathbf{x}_0 = 0 \ .$$

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However, in general the noise and the state are not uncorrelated. Using Eq.(2) repeatedly, we get

$$\begin{aligned}\langle N_k \mathbf{x}_k\rangle &= \mathbf{R}\langle N_k \mathbf{x}_{k-1}\rangle + \begin{pmatrix}0\\1\end{pmatrix}\langle N_k N_{k-1}\rangle \\ &= \mathbf{R}^2\langle N_k \mathbf{x}_{k-2}\rangle + \mathbf{R}\begin{pmatrix}0\\1\end{pmatrix}\langle N_k N_{k-2}\rangle + \begin{pmatrix}0\\1\end{pmatrix}\langle N_k N_{k-1}\rangle \\ &= \sum_{n=0}^{k-1}\left[\mathbf{R}^n\begin{pmatrix}0\\1\end{pmatrix}\right]\langle N_k N_{k-1-n}\rangle \ .\end{aligned}$$

We note in passing that $\mathbf{R}$ is just a rotation matrix, so that

$$\mathbf{R}^n = \begin{pmatrix}\cos 2\pi n\nu & \sin 2\pi n\nu \\ -\sin 2\pi n\nu & \cos 2\pi n\nu\end{pmatrix} \ .$$

Now we can go back to Eq.(4) and rewrite Eq.(6) as follows.

$$\langle\epsilon_k\rangle = \langle\epsilon_{k-1}\rangle + \frac{2\pi}{\beta}(0\ 1)\mathbf{R}\langle N_{k-1}\mathbf{x}_{k-1}\rangle + \frac{\pi}{\beta}\langle N_{k-1}^2\rangle \tag{8}$$

Using our previous result, we rewrite the new, second term.

$$\begin{aligned}2^{\text{nd}}\text{ term} &= \frac{2\pi}{\beta}\sum_{n=1}^{k-1}\left[(0\ 1)\mathbf{R}^n\begin{pmatrix}0\\1\end{pmatrix}\right]\langle N_{k-1}N_{k-1-n}\rangle \\ &= \frac{2\pi}{\beta}\sum_{n=1}^{k-1}\cos(2\pi n\nu)\langle N_{k-1}N_{k-1-n}\rangle \\ &= \frac{2\pi}{\beta}\sum_{n=1}^{k-1}\cos(2\pi n\nu)\Phi_n\end{aligned}$$

Plugging this back into Eq.(8),

$$\langle\epsilon_k\rangle = \langle\epsilon_{k-1}\rangle + \frac{2\pi}{\beta}\sum_{n=1}^{k-1}\cos(2\pi n\nu)\Phi_n + \frac{\pi}{\beta}\Phi_0$$

The difference, $\langle\epsilon_k\rangle - \langle\epsilon_{k-1}\rangle$, now depends on k. However, we can take the limit,

$$\begin{aligned}\Delta\epsilon_\infty &= \lim_{k\to\infty}[\langle\epsilon_k\rangle - \langle\epsilon_{k-1}\rangle] \\ &= \frac{\pi}{\beta}\Phi_0 + \frac{2\pi}{\beta}\sum_{n=1}^{\infty}\cos(2\pi n\nu)\Phi_n \\ &= \frac{\pi}{\beta}\sum_{n=-\infty}^{\infty}\cos(2\pi n\nu)\Phi_n\end{aligned} \tag{9}$$

The limiting expected emittance growth rate would then be $r = f \times \Delta\epsilon_\infty$.

As an example, consider a sinusoidally varying kick,

$$N_k = A\sin(k\omega\tau + \varphi) \ ,$$

where A and ω are constant parameters, $\tau = 1/f$ is the revolution period through the ring, and φ is a random variable distributed on $[0, 2\pi)$ according to a probability measure $d\mu(\varphi)$. (This example is slightly illegitimate, but let us continue.) The ensemble of noise signals is indexed by φ, and each member of the ensemble varies sinusoidally. Of $d\mu(\varphi)$ we demand only that it have no first or second harmonics:

$$\int d\mu(\varphi)e^{i\varphi} = \int d\mu(\varphi)e^{2i\varphi} = 0 \ .$$

Then it is obvious that $\langle N_k\rangle = 0$, as required, while

$$\begin{aligned}\Phi_{k-m} &= \langle N_k N_m\rangle = A^2\int d\mu(\varphi)\sin(k\omega\tau+\varphi)\sin(m\omega\tau+\varphi) \\ &= A^2\int d\mu(\varphi)\frac{1}{2}\{\cos[(k-m)\omega\tau] - \cos[(k+m)\omega\tau + 2\varphi]\} \\ &= \frac{1}{2}A^2\cos[(k-m)\omega\tau] \ .\end{aligned}$$

We now evaluate the discrete cosine transform.

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\cos(2\pi n\nu)\Phi_n &= \frac{1}{2}A^2\sum_{n=-\infty}^{\infty}\cos(2\pi n\nu)\cos(n\omega\tau) \\ &= \frac{1}{4}A^2\sum_{n=-\infty}^{\infty}\cos[n(2\pi\nu+\omega\tau)] + \cos[n(2\pi\nu-\omega\tau)] \\ &= \frac{\pi}{2}A^2 \times [\delta(2\pi\nu+\omega\tau) + \delta(2\pi\nu-\omega\tau)]\end{aligned}$$

That we get delta functions reflects the fact that the autocorrelation functions Φ_n does not vanish for large n. If we want to avoid generalized functions, Φ_n should be tempered in some manner; the delta functions would then revert to regular functions sharply peaked at the zero of their arguments. The important point here is that the result is zero for non-vanishing arguments of the delta function.

If we associate, as in Eq.(7), the noise with a fluctuating dipole field, then the the limiting rate of emittance growth is expressed,

$$r = \frac{1}{2}f\pi^2\beta\left(\frac{Bl}{|B\rho|}\right)^2\left(\frac{\delta B}{B}\right)^2_{max} \times [\delta(2\pi\nu+\omega\tau) + \delta(2\pi\nu-\omega\tau)] \tag{10}$$

3 Distributed kicks.

Now consider a series of kicks distributed about the ring. We shall write this in terms of a kick function $N(\phi)$,

$$N(\phi) = \sum_{k=-\infty}^{\infty} N_k\delta(\phi-\phi_k), \tag{11}$$

whose argument is $\phi = \psi/\nu$. (This change of variables from θ to ϕ is made only for convenience. The end result would be the same if we stayed with θ, but the intermediate steps would be more cumbersome. Of course, for a two degree of freedom calculation we would have to use θ and the matrix formulation.) The angles ϕ_k are completely arbitrary; they need not be equally spaced around the ring. $N(\phi)$ is formally written as an infinite sum, but it will be finite if all but a finite number of N_k's vanish. In such a case, $N(\phi)$ will have bounded support.

In this section we shall express the state as a complex variable. To this end, we define

$$z \equiv p + ix = \sqrt{2I\beta}\exp[i(\tilde\psi + \delta)] \ .$$

(If we were to quantize this system, $-iz$ would become a creation operator.) For unperturbed motion, z_o, we have: $d\delta/d\theta = \nu$, $\tilde\psi + \delta = \psi + \delta|_{\theta=0}$, and I is a constant of the motion. Thus,

$$\begin{aligned}\frac{d}{d\theta}(z_o/\sqrt{\beta}) &= i\frac{d\psi}{d\theta}(z_o/\sqrt{\beta}) \\ &= i\nu\frac{d\phi}{d\theta}(z_o/\sqrt{\beta}) \\ \frac{d}{d\phi}(z_o/\sqrt{\beta}) &= i\nu(z_o/\sqrt{\beta}) \ .\end{aligned} \tag{12}$$

When the particle is kicked, the state changes instantaneously according to

$$\Delta\mathbf{x}(\phi_k)|_{noise} = \begin{pmatrix}0\\1\end{pmatrix}N_k \ ,$$

in direct analogy to Eq.(2). We write this in terms of $N(\phi)$,

$$d\mathbf{x}/d\phi|_{\text{noise}} = \begin{pmatrix}0\\1\end{pmatrix}N(\phi) \ ,$$

and finally as a dynamic for z,

$$\left.\frac{dz}{d\phi}\right|_{\text{noise}} = N(\phi)$$

We add this to the unperturbed motion of Eq.(12) to get the full dynamical system.

$$\left(\frac{d}{d\phi} - i\nu\right)(z/\sqrt{\beta}) = N/\sqrt{\beta} \tag{13}$$

The Green's function for the linear operator on the left is

$$G(\phi-\phi') = \Theta(\phi-\phi')e^{i\nu(\phi-\phi')} \ , \tag{14}$$

where Θ is the usual Heaviside step function. (Note that we do not want a periodic Green's function. The boundary condition is that $G(\phi-\phi') = 0$ for $\phi < \phi'$.) The solution to Eq.(13) is then written,

$$\begin{aligned}z/\sqrt{\beta} &= z_o/\sqrt{\beta} + \int_{-\infty}^{\infty} d\phi' G(\phi-\phi')N(\phi')/\sqrt{\beta(\phi')} \\ &= z_o/\sqrt{\beta} + \int_{-\infty}^{\phi} d\phi' e^{i\nu(\phi-\phi')}N(\phi')/\sqrt{\beta(\phi')}\end{aligned}$$

Note that $\lim_{\phi\to-\infty} z = z_o$. This is suggestive of quantum scattering theory. Let us define the *in* and *out* states of this problem,

$$\begin{aligned}z_{out}/\sqrt{\beta^*} &= \lim_{\phi\to+\infty} ze^{-i\nu\phi}/\sqrt{\beta} \\ z_{in}/\sqrt{\beta^*} &= \lim_{\phi\to-\infty} z_o e^{-i\nu\phi}/\sqrt{\beta} \ ,\end{aligned}$$

where β^* serves only to carry the units and to provide an appropriate scale. Then we have the association,

$$z_{out} = z_{in} + \sqrt{\beta^*}\int_{-\infty}^{\infty} e^{-i\nu\phi'} N(\phi')/\sqrt{\beta(\phi')} \qquad (15)$$

Of course, this has meaning only if $N(\phi)$ has bounded support.

The single-particle "emittance" is

$$\begin{aligned}\epsilon = \frac{\pi}{\beta}\mathbf{x}^T\mathbf{x} &= \frac{\pi}{\beta}|z|^2 \\ &= \pi\left| z_o e^{-i\nu\phi}/\sqrt{\beta} + \int_{-\infty}^{\phi} d\phi' e^{-i\nu\phi'} N(\phi')/\sqrt{\beta(\phi')}\right|^2\end{aligned}$$

The limiting result depends on the Fourier transform, relative to ϕ, of $N/\sqrt{\beta}$ at the tune value, ν. Suppose that $N/\sqrt{\beta} = A\cos(\nu\phi)$ for the duration $\phi \in [0, \phi_{max}]$ and vanishes everywhere else. Then, for large ϕ_{max} we have the asymptotic expression,

$$\epsilon_{out} \approx \frac{\pi A^2}{4}\phi_{max}^2 .$$

This is reasonable. Consider what happens to a harmonic oscillator which is kicked in phase every time it passes the origin: its momentum increases linearly, and therefore its energy increases quadratically, with the number of kicks.

Now let us once again make N a random function and evaluate the ensemble average over all possible noise histories. If we again assume a zero-mean process, $\forall\phi : \langle N(\phi)\rangle = 0$, then,

$$\begin{aligned}\langle\epsilon\rangle_{out} &= \epsilon_{in} + \pi\int\int \frac{d\phi' d\phi''}{\sqrt{\beta(\phi')\beta(\phi'')}} \exp[\, i\nu(\phi' - \phi'')\,] \times \langle N(\phi')N(\phi'')\rangle \\ &= \epsilon_{in} + \pi\int\int \frac{d\phi' d\phi''}{\sqrt{\beta(\phi')\beta(\phi'')}} \cos[\,\nu(\phi' - \phi'')\,] \times \langle N(\phi')N(\phi'')\rangle\end{aligned}$$

In particular, if $\langle N(\phi')N(\phi'')\rangle$ vanished except when ϕ' and ϕ'' represent the same location in the ring,

$$\langle N(\phi')N(\phi'')\rangle = \sum_{n=-\infty}^{\infty} K_n(\phi')\delta(\phi' - \phi'' - 2\pi n) \;,$$

then this becomes,

$$\langle\epsilon\rangle_{out} = \epsilon_{in} + \pi\sum_{n=-\infty}^{\infty}\cos[\,2\pi n\nu\,]\int_{-\infty}^{\infty}\frac{d\phi}{\beta(\phi)}K_n(\phi)$$

Finally, if N operates only at one point in the ring, say ϕ_0, and if the process is stationary, then we identify

$$K_n(\phi) = \Phi_n\sum_k \delta(\phi - \phi_0 - 2\pi k) \;,$$

and recapture the result of Section 2:

$$\langle\epsilon\rangle_{out} = \epsilon_{in} + \left(\sum_m 1\right)\left(\frac{\pi}{\beta(\phi_0)}\sum_{n=-\infty}^{\infty}\cos(2\pi n\nu)\Phi_n\right)$$

The extra infinite sum appears because we are here not calculating the rate of emittance growth but the final emittance, which must be infinite for a stationary process.

Making a switch from ensemble averages to time averages, we note that the quantity Φ_n is, with probability 1 as $T \to \infty$, the time autocorrelation function corresponding to the stochastic function $N(t)$ and is, according to the Wiener-Khintschin theorem, the Fourier transform of the power spectrum of N.

$$\Phi(\tau) = lim_{T\to\infty}\frac{1}{T}\int_0^T dt\, N(t)N(t+\tau) = \int d\omega\, e^{i\omega\tau}P(\omega)$$

We now proceed to calculate M_1, as in Eq.(9).

$$\begin{aligned}M_1 &= \Delta I/T = \beta f_0\sum_{n=0}^{\infty}\cos n\mu\,\Phi(n/f_0) \\ &= \frac{1}{2}\beta f_0\sum_n\int d\omega\left(e^{in(\omega/f_0\pm\mu)} + c.c.\right)P(\omega) \\ &= \frac{1}{2}\beta f_0^2\sum_{p,\pm}2\pi P_\omega(2\pi p f_0 \pm \mu f_0) \\ &= \beta f_0^2\sum_{\text{positive frequencies}} P_f(nf_0 \pm \nu f_0)\end{aligned}$$

Then, the first moment, like the second, only attains a non-zero value if the power spectrum is non-zero at the Schottky lines. If P_f contains many revolution frequencies in its bandwidth, $M_1 \approx 2\beta f_0\int P_f df$; that is, the total power is all that matters.

Let us apply this to a low frequency process,

$$N(t) = \theta_1\cos(\Omega t + \phi(t)) \;,$$

where θ_1 is constant, Ω is a frequency below the lowest Schottky line, and ϕ is a zero-mean, random function of time (a) which is small, $\phi \ll 1$, and (b) has autocorrelation function $C_\phi(\tau)$. It is easily verified that

$$\Phi(\tau) = \frac{1}{2}\theta_1^2[1 + C_\phi(\tau)]\cos\Omega\tau$$

If we now take the Fourier transform of this to get the power spectrum, in addition to delta function terms, as in Eq.(10), we obtain a term

$$\frac{1}{4\pi}\int d\tau C_\phi(\tau)[e^{-i(\omega+\Omega)\tau} + c.c.] \propto P_\phi(\omega \pm \Omega)$$

Then the heating takes place at sideband frequencies $f_0(n \pm \nu) \pm \Omega$ in the power spectrum of ϕ. We would expect the bandwidth of the ϕ noise to be less that Ω, so for low frequencies Ω we would not expect to get heating from this type of phase noise. A similar conclusion pertains to random changes in the amplitude θ_1.

4 Concluding comments.

The formalisms used in the two preceding sections are interchangeable. We could just as easily have written

$$z_k = e^{i2\pi\nu}z_{k-1} + N_{k-1}$$

for Eq.(2) or used a matrix Green's function

$$\begin{aligned}\mathbf{G}(\phi - \phi') &= \Theta(\phi - \phi')\exp[\,\nu(\phi - \phi')\mathbf{J}\,] \\ \mathbf{J} &= \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\end{aligned}$$

in place of Eq.(14). We have recorded derivations in both formalisms for purposes of illustration.

The two results, nonetheless, look very different. In the first section we clearly have an emittance that is growing indefinitely with time, while in the second the emittance attains a final value. (See Eq.(15), for example.) The difference is that in Section 2 we assumed a stationary process, which therefore continues indefinitely, while the evaluation of Fourier harmonics in Section 3 required a tacit assumption of a noise functions which damped out with time. Thus, the infinite series in Eq.(11) is really finite; all but a finite number N_k's vanish. Nonetheless, while the noise function is non-zero, the emittance grows quadratically with time, whereas in the case of equally spaced, random kicks the asymptotic growth rate of the *expected emittance from an ensemble of noise histories* is a constant.

It turns out that a close variant of Eq.(9) was derived but not published by Gerry Dugan several years ago; these results probably exist in the desk drawers of a number of other people as well. We have treated only single particle motion here. A treatment of true emittance growth requires considering motion of the centroid of a bunch and the mixing of particles within the bunch. Merminga, Mane, and Edwards have demonstrated the equivalence of various approaches to calculating the decoherence of a beam.[3] Mane has generalized our formalism by adding a damping term as an approximate way of modelling the motion of the centroid with detuning and thereby has developed predictions which compare favorably with emittance growth measurements in the Tevatron.[2]

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