Dipole Fringe Field Optics in TRANSPORT

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Abstract

Analytical solutions to the equations of motion of a charged particle through the extended fringing field of a dipole magnet are obtained in the formalism of the computer code TRANSPORT. They are represented by first-, second-, and third-order transfer matrices which contain integral form factors of the field. A straight inclined boundary is considered.

1 Introduction

The optical effects of the field boundary of a dipole magnet have been described to the second order for the sharp-cutoff approximation [1] and to the first order for the extended fringing fields [2,3]. In the third order, the sharp-cutoff approximation produces infinities in the matrix elements [4]. To obtain the full third-order solution, one must consider the extended nature of the field.

In this paper we describe the calculation of the transfer matrices for the fringing field of a straight dipole boundary. The matrix elements are given by the coefficients in the Taylor expansion,

$$X_a^f = \sum_b R_{ab} X_b + \sum_b \sum_c T_{abc} X_b X_c + \sum_b \sum_c \sum_d U_{abcd} X_b X_c X_d + \cdots \quad (1)$$

where $\mathbf{X}$ is the usual 6-vector of the TRANSPORT [5] coordinates, $\mathbf{X} = (x, x', y, y', l, \delta)$.

It is convenient to characterize the extent of the transition field region with a dimensionless parameter $\epsilon = d/\rho$, where ρ is the inside bending radius and d is the vertical separation of the poles. We assume that $\epsilon \ll 1$ and derive the matrices R, T, U to $O(\epsilon)$. In this approximation, we find that the matrix elements depend on certain dimensionless line integrals. There are two such integrals present in the second order and six more in the third order. Also, we can identify the term which leads to a divergence in the sharp-cutoff limit.

The procedure to obtain transfer matrices without using the ϵ expansion is outlined in [6]. It is based on the Lie algebraic approach [7] and the connection formulas [8] between the canonical Hamiltonian variables and the TRANSPORT coordinates. The complete description of the problem as well as the explicit list of all the matrix elements can be found in [9].

2 Mathematical Formulation

We consider the entrance of the bending magnet shown in Fig. 1. The coordinate s measures the distance from the the effective field boundary. The coordinate system (s, u, y) is rotated clockwise around the y-axis with respect to the reference system (z, x, y). The rotation angle β is taken to be positive.

The net effect of the fringe field of an inclined boundary can be mathematically represented by a fictitious optical element of zero thickness, located at the reference plane [1,3]. The transfer matrix for such a lens is given by a product of three transformations,

$$\mathcal{M}^{0\mapsto f} = \mathcal{M}^{2\mapsto f} \mathcal{M}^{1\mapsto 2} \mathcal{M}^{0\mapsto 1} \quad (2)$$

where

1. $\mathcal{M}^{0\mapsto 1}$ is a transformation from the reference plane to the beginning of the fringe region through the pure drift field;
2. $\mathcal{M}^{1\mapsto 2}$ is the transformation through the fringe region;
3. $\mathcal{M}^{2\mapsto f}$ is the transformation from the end of the fringe region back to the reference plane through the pure bend field.

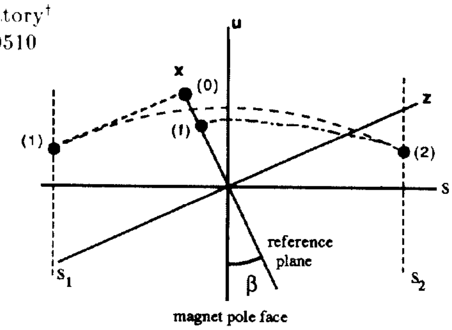

Figure 1: Midplane geometry. Reference plane is normal to the design trajectory. Angle β is taken to be positive. Axis y points out of the paper.

The field in the air gap makes a gradual transition from the longitudinally uniform interior field B_0 to the field-free region external to the magnet. We assume the midplane field depends on the distance s only, i. e. we neglect the effects connected to the finite width of the magnet. We define

$$h(s) = \frac{B_y(y = 0)}{B_0} \quad (3)$$

3 Derivation of Matrix Elements

3.1 Transformation Variables

We can write the relationship between the two sets of variables,

$$\begin{aligned} u &= z\sin\beta + x\cos\beta \\ s &= z\cos\beta - x\sin\beta \\ \dot{u} &= \frac{du/dz}{ds/dz} = \frac{x' + \tan\beta}{1 - x'\tan\beta} \\ \dot{y} &= \frac{dy/dz}{ds/dz} = \frac{y'\sec\beta}{1 - x'\tan\beta} \end{aligned} \quad (4)$$

where the dot denotes d/ds and the prime denotes d/dz. Anticipating future expansions, we define a new variable w as the deviation from the reference trajectory,

$$\left.\begin{aligned} u(s) &= w(s) + \Delta(s) \\ \dot{u}(s) &= \dot{w}(s) + \dot{\Delta}(s) \end{aligned}\right\}$$

Let us next scale the variables as follows,

$$\begin{aligned} s &\to s/d \\ \zeta &\to \zeta/\rho \end{aligned}$$

where a generic ζ denotes any of x, y, z, u, w, Δ. From now on, we will use the explicit symbol d/ds to denote the dimensionless derivative

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and the dot to denote the differentiation with respect to the original unscaled variable s.

To obtain the matrix elements, we must perform the following transformations,

$$\begin{pmatrix} z=0 \\ x_0 \\ x'_0 \\ y_0 \\ y'_0 \end{pmatrix} \xrightarrow{\mathcal{M}^{0\to 1}} \begin{pmatrix} s=s_1 \\ w_1 \\ \dot{w}_1 \\ y_1 \\ \dot{y}_1 \end{pmatrix} \xrightarrow{\mathcal{M}^{1\to 2}} \begin{pmatrix} s=s_2 \\ w_2 \\ \dot{w}_2 \\ y_2 \\ \dot{y}_2 \end{pmatrix} \xrightarrow{\mathcal{M}^{2\to f}} \begin{pmatrix} z=0 \\ x^f \\ x'^f \\ y^f \\ y'^f \end{pmatrix}$$

3.2 Drift Region Transformation

In the drift region, $\mathbf{B}=0$ and the equations of motion are simply

$$x''=y''=0 \tag{5}$$

Using Eq. 5 and expanding to the third order in the initial conditions (x_0, x'_0, y_0, y'_0), we get the drift map,

$$\begin{aligned} w_1 &= \left(x_0+\epsilon s_1 x'_0 \sec\beta\right)\sec\beta+\left(x_0 x'_0+\epsilon s_1 {x'_0}^2 \sec\beta\right)\sec\beta\tan\beta \\ &\quad +\left(x_0 {x'_0}^2+\epsilon s_1 {x'_0}^3 \sec\beta\right)\sec\beta\tan^2\beta \\ \dot{w}_1 &= x'_0\sec^2\beta+{x'_0}^2\sec^2\beta\tan\beta+{x'_0}^3\sec^2\beta\tan^2\beta \\ y_1 &= \left(y_0+\epsilon s_1 y'_0\sec\beta\right)+\left(x_0 y'_0+\epsilon s_1 x'_0 y'_0\sec\beta\right)\tan\beta \\ &\quad +\left(x_0 x'_0 y'_0+\epsilon s_1 {x'_0}^2 y'_0\sec\beta\right)\tan^2\beta \\ \dot{y}_1 &= y'_0\sec\beta+x'_0 y'_0\sec\beta\tan\beta+{x'_0}^2 y'_0\sec\beta\tan^2\beta \end{aligned} \tag{6}$$

The reference trajectory's coordinates at $s=s_1$ are given by

$$\begin{aligned} \Delta_1 &= \epsilon s_1\tan\beta \\ \dot{\Delta}_1 &= \tan\beta \end{aligned}$$

3.3 Fringe Region Transformation

Using Maxwell's equations and the midplane symmetry we can expand the magnetic field components around $b(s)=B_y(s,u,y=0)$:

$$\begin{aligned} B_s(s,u,y) &= \dot{b}y+\cdots \\ B_u(s,u,y) &= 0 \\ B_y(s,u,y) &= b-\frac{1}{2}\ddot{b}y^2+\cdots \end{aligned} \tag{7}$$

From the above expansions, we can write the equations of motion as follows,

$$\begin{aligned} \ddot{u} &= -T\left[f\left(1+\dot{u}^2\right)+g\dot{u}\dot{y}\right] \\ \ddot{y} &= -T\left[g\left(1+\dot{y}^2\right)+f\dot{u}\dot{y}\right] \end{aligned} \tag{8}$$

where

$$T=\left(1+\dot{u}^2+\dot{y}^2\right)^{\frac{1}{2}}$$

$$f=\frac{1}{\rho(1+\delta)}\left(h-\dot{h}y\dot{y}-\frac{1}{2}\ddot{h}y^2\right)$$

$$g=\frac{1}{\rho(1+\delta)}\dot{h}uy$$

We obtain the equation for Δ by putting $u=\Delta$, $y=\dot{y}=\delta=0$ in Eq. 8:

$$\ddot{\Delta}=-\frac{h}{\rho}(1+\dot{\Delta}^2)^{\frac{3}{2}} \tag{9}$$

The above equation can readily be solved by iteration,

$$\dot{\Delta}(s)=\tan\beta-\epsilon\sec^3\beta\int_{s_1}^{s}h(s')ds'+O\left(\epsilon^2\right) \tag{10}$$

We expand Eq. 8 to the third order in w, $\dot{w}$, y, $\dot{y}$, and δ. The result can be written in the form of the dimensionless equations as follows,

$$\begin{aligned} \frac{dw}{ds} &= \epsilon\dot{w} \\ \frac{d\dot{w}}{ds} &= \epsilon h\left(3\dot{w}\Gamma_1^2\Gamma_2-\delta\Gamma_1^3\right)+\left[\frac{1}{2\epsilon}\frac{d^2h}{ds^2}y^2\Gamma_1^3+\frac{dh}{ds}y\dot{y}\Gamma_1\right. \\ &\quad \left.-\epsilon h\left(\frac{3}{2}\dot{w}^2\Gamma_3-3\dot{w}\delta\Gamma_1^2\Gamma_2+\frac{1}{2}\dot{y}^2\Gamma_1+\delta^2\Gamma_1^3\right)\right] \\ &\quad +\left[\frac{1}{2\epsilon}\frac{d^2h}{ds^2}\left(3\dot{w}y^2\Gamma_1^2\Gamma_2-y^2\delta\Gamma_1^3\right)+\frac{dh}{ds}\left(\dot{w}y\dot{y}\Gamma_2-y\dot{y}\delta\Gamma_1\right)\right. \\ &\quad -\epsilon h\left(\frac{1}{2}\dot{w}^3\Gamma_4-\frac{3}{2}\dot{w}^2\delta\Gamma_3+\frac{1}{2}\dot{w}\dot{y}^2\Gamma_2\right. \\ &\quad \left.\left.+3\dot{w}\delta^2\Gamma_1^2\Gamma_2-\frac{1}{2}\dot{y}^2\delta\Gamma_1-\delta^3\Gamma_1^3\right)\right] \\ \frac{dy}{ds} &= \epsilon\dot{y} \\ \frac{d\dot{y}}{ds} &= -\frac{d}{ds}(hy)\left[\Gamma_1^2\Gamma_2+\left(\dot{w}\Gamma_3-\delta\Gamma_1^2\Gamma_2\right)+\right. \\ &\quad \left.\left(\frac{1}{2}\dot{w}^2\Gamma_4-\dot{w}\delta\Gamma_3+\frac{1}{2}\dot{y}^2\Gamma_2+\delta^2\Gamma_1^2\Gamma_2\right)\right]+\frac{1}{2\epsilon}\frac{d^2h}{ds^2}y^2\dot{y}\Gamma_1^2\Gamma_2 \end{aligned} \tag{11}$$

where

$$\Gamma_1=\left(1+\dot{\Delta}^2\right)^{\frac{1}{2}}\quad \Gamma_2=\frac{\dot{\Delta}}{(1+\dot{\Delta}^2)^{\frac{1}{2}}}\quad \Gamma_3=\frac{(1+2\dot{\Delta}^2)}{(1+\dot{\Delta}^2)^{\frac{1}{2}}}\quad \Gamma_4=\frac{\dot{\Delta}(3+2\dot{\Delta}^2)}{(1+\dot{\Delta}^2)^{\frac{3}{2}}}$$

The first-order solution to Eq. 11 is well known [2]. We will obtain the nonlinear part of the solution using the order-by-order method similar to the Green's function integration employed for the case of an ideal magnet [10,11].

3.3.1 Second Order Coefficients

The most general second-order solution to Eq. 11 is given by

$$\begin{aligned} w &= R_{11}w_1+R_{12}\dot{w}_1+R_{16}\delta \\ &\quad +T_{122}\dot{w}_1^2+T_{126}\dot{w}_1\delta+T_{133}y_1^2+T_{134}y_1\dot{y}_1+T_{144}\dot{y}_1^2+T_{166}\delta^2 \\ \dot{w} &= R_{22}\dot{w}_1+R_{26}\delta \\ &\quad +T_{222}\dot{w}_1^2+T_{226}\dot{w}_1\delta+T_{233}y_1^2+T_{234}y_1\dot{y}_1+T_{244}\dot{y}_1^2+T_{266}\delta^2 \\ y &= R_{33}y_1+R_{34}\dot{y}_1 \\ &\quad +T_{323}\dot{w}_1y_1+T_{324}\dot{w}_1\dot{y}_1+T_{336}y_1\delta+T_{346}\dot{y}_1\delta \\ \dot{y} &= R_{43}y_1+R_{44}\dot{y}_1 \\ &\quad +T_{423}\dot{w}_1y_1+T_{424}\dot{w}_1\dot{y}_1+T_{436}y_1\delta+T_{446}\dot{y}_1\delta \end{aligned} \tag{12}$$

Putting Eq. 12 into Eq. 11 and equating the same second-order terms, we get the equations for T_{ijk}'s,

$$\begin{aligned} \frac{d}{ds}T_{1ij} &= \epsilon T_{2ij} \\ \frac{d}{ds}T_{2ij} &= -3\epsilon\Gamma_1^2\Gamma_2 hT_{2ij}+f_{2ij} \\ \frac{d}{ds}T_{3ij} &= \epsilon T_{4ij} \\ \frac{d}{ds}T_{4ij} &= -\Gamma_1^2\Gamma_2\frac{d}{ds}(hT_{3ij})+f_{4ij} \end{aligned} \tag{13}$$

The functions f_{2ij} and f_{4ij} are given in Table 1. Eq. 13 can be solved by iteration to a desired order in ϵ. We get the following expansion series,

Table 1: Driving Terms For Second-Order Matrix Elements

T_{ijk}	f_{ijk}
T_{222}	$3\epsilon hR_{22}^2\Gamma_3/2$
T_{226}	$3\epsilon h(R_{22}\Gamma_1^2\Gamma_2-R_{22}R_{26}\Gamma_3)$
T_{233}	$\left(d^2h/ds^2R_{33}^2\Gamma_1^3/\epsilon+2dh/dsR_{33}R_{43}\Gamma_1-\epsilon hR_{43}^2\Gamma_1\right)/2$
T_{234}	$\left(d^2h/ds^2R_{33}R_{34}\Gamma_1^3\epsilon+dh/ds(R_{33}R_{44}+R_{34}R_{43})\Gamma_1-\epsilon hR_{43}R_{44}\Gamma_1\right)$
T_{244}	$\left(d^2h/ds^2R_{34}^2\Gamma_1^3/\epsilon+2dh/dsR_{34}R_{44}\Gamma_1-\epsilon hR_{44}^2\Gamma_1\right)/2$
T_{266}	$-\epsilon h(3R_{26}^2\Gamma_3-6R_{26}\Gamma_1^2\Gamma_2+2\Gamma_1^3)/2$
T_{423}	$-dh/dsR_{22}R_{33}\Gamma_3-\epsilon hR_{22}R_{43}\Gamma_3$
T_{424}	$-dh/dsR_{22}R_{34}\Gamma_3-\epsilon hR_{22}R_{44}\Gamma_3$
T_{436}	$dh/ds(R_{33}\Gamma_1^2\Gamma_2-R_{26}R_{33}\Gamma_3)+\epsilon h(R_{43}\Gamma_1^2\Gamma_2-R_{26}R_{43}\Gamma_3)$
T_{446}	$dh/ds(R_{34}\Gamma_1^2\Gamma_2-R_{26}R_{34}\Gamma_3)+\epsilon h(R_{44}\Gamma_1^2\Gamma_2-R_{26}R_{44}\Gamma_3)$

$$\begin{aligned} T_{1ij}(s) &= \epsilon\int_{s_1}^{s}T_{2ij}(s')ds' \\ T_{2ij}(s) &= \int_{s_1}^{s}f_{2ij}(s')ds' \\ &\quad -3\epsilon\int_{s_1}^{s}\Gamma_1^2(s')\Gamma_2(s')h(s')\int_{s_1}^{s'}f_{2ij}(s'')ds''ds'+\cdots \\ T_{3ij}(s) &= \epsilon\int_{s_1}^{s}\int_{s_1}^{s'}f_{4ij}(s'')ds''ds'+\cdots \\ T_{4ij}(s) &= \int_{s_1}^{s}f_{4ij}(s')ds'-\Gamma_1^2(s)\Gamma_2(s)h(s)T_{3ij}(s)+\cdots \end{aligned} \tag{14}$$

3.3.2 Third Order Coefficients

We add the third order terms with coefficients U_{ijkl} to the expansions in Eq. 12. Then, we put Eq. 12 into Eq. 11 and equate the same third-order terms. We get

$$
\begin{aligned}
\frac{d}{ds}U_{1ijk} &= \epsilon U_{2ijk} \\
\frac{d}{ds}U_{2ijk} &= -3\epsilon\Gamma_1^2\Gamma_2 h U_{2ijk} + g_{2ijk} \\
\frac{d}{ds}U_{3ijk} &= \epsilon U_{4ijk} \\
\frac{d}{ds}U_{4ijk} &= -\Gamma_1^2\Gamma_2 \frac{d}{ds}\left(hU_{3ijk}\right) + g_{4ijk}
\end{aligned} \tag{15}
$$

The functions g_{2ijk} and g_{4ijk} depend on Γ_n's, R_{ij}'s and T_{ijk}'s, and are given in [9]. Eq. 15 can also be solved by iteration; matrix elements U_{ijkl} have the same iteration series as do the elements T_{ijk} in Eq. 14.

3.4 Bend Region Transformation

In the pure bend region, $\mathbf{B} = B_0\hat{\mathbf{y}}$ and the equations of motion can be written as follows,

$$
\begin{aligned}
x'' &= -\frac{1}{\rho(1+\delta)}\left(1+x'^2\right)\left(1+x'^2+y'^2\right)^{\frac{1}{2}} \\
y'' &= -\frac{1}{\rho(1+\delta)}\, x'\, y'\left(1+x'^2+y'^2\right)^{\frac{1}{2}}
\end{aligned} \tag{16}
$$

We can solve Eq. 16 exactly for x' and y',

$$
\begin{aligned}
x' &= \frac{x_2'(1+\delta) - \tau(z-z_2)}{\left[(1+\delta)^2 + 2x_2'(1+\delta)\tau(z-z_2) - \tau^2(z-z_2)^2\right]^{\frac{1}{2}}} \\
y' &= \frac{y_2'}{\left[(1+\delta)^2 + 2x_2'(1+\delta)\tau(z-z_2) - \tau^2(z-z_2)^2\right]^{\frac{1}{2}}}
\end{aligned} \tag{17}
$$

where

$$
\tau = \sqrt{1 + {x_2'}^2 + {y_2'}^2}
$$

The above equations can be integrated to obtain x and y. We can then put $z = 0$ and expand to the third order in $(w_2, \dot{w}_2, y_2, \dot{y}_2, \delta)$ using Eq. 5 and Eq. 10.

4 Nonlinear Solutions

4.1 Second Order Matrix Elements

The second-order solution contains two integral form factors,

$$
\begin{aligned}
I_2 &= \int_{-\infty}^{\infty} ds\,[1 - h(s)]\,h(s) \\
I_3 &= \int_{-\infty}^{\infty} ds\,[1 - h(s)]\,h^2(s)
\end{aligned} \tag{18}
$$

The integrands in Eq. 18 go rapidly to zero at both sides of the integration limits; this fact allows to remove the uncertainty in the extent of the fringe region and gives a practical way for the integrals' evaluation. The 10 non-zero terms are given below:

$$
\begin{aligned}
T_{111} &= \frac{\tan^2\beta}{2} + \cdots \\
T_{133} &= \frac{\sec^2\beta}{2} - \epsilon I_2 \frac{\sin\beta(5 - \sin^2\beta)}{2\cos^4\beta} + \cdots \\
T_{212} &= \tan^2\beta + \cdots \\
T_{216} &= -\tan\beta + \cdots \\
T_{233} &= \frac{\sin\beta(1+\sin^2\beta)}{2\cos^3\beta} \\
&\quad - \epsilon\frac{\sin^2\beta}{\cos^5\beta}\left[I_2(5 - \cos^4\beta) - I_3\frac{\cos^2\beta}{2}\right] + \cdots \\
T_{234} &= -\tan^2\beta - \epsilon I_2 \frac{\sin\beta(8+\cos^2\beta)}{\cos^4\beta} + \cdots \\
T_{313} &= \tan^2\beta - \epsilon I_2 \frac{\sin\beta(1+\sin^2\beta)}{\cos^4\beta} + \cdots \\
T_{414} &= -\tan^2\beta + \epsilon I_2 \frac{\sin\beta(1+\sin^2\beta)}{\cos^4\beta} + \cdots \\
T_{423} &= -\sec^2\beta + \epsilon I_2 \frac{\sin\beta(5+\sin^2\beta)}{\cos^4\beta} + \cdots \\
T_{436} &= \tan\beta - \epsilon I_2 \frac{2(1+\sin^2\beta)}{\cos^3\beta} + \cdots
\end{aligned}
$$

The above matrix elements are in perfect agreement with the known sharp-cutoff approximation results [1] ($\epsilon = 0$).

4.2 Third Order Matrix Elements

The third-order solution has six more integral form factors, all which contain the square of the field derivative,

$$
\begin{aligned}
J_1 &= \int_{-\infty}^{+\infty} ds \left(\frac{dh(s)}{ds}\right)^2 \\
J_2 &= \int_{-\infty}^{+\infty} ds \left(\frac{dh(s)}{ds}\right)^2 s \\
J_3 &= \int_{-\infty}^{+\infty} ds \left(\frac{dh(s)}{ds}\right)^2 s^2 \\
J_4 &= \int_{-\infty}^{+\infty} ds \left(\frac{dh(s)}{ds}\right)^2 \int_{-\infty}^{s} ds' h(s') \\
J_5 &= \int_{-\infty}^{+\infty} ds \left(\frac{dh(s)}{ds}\right)^2 s \int_{-\infty}^{s} ds' h(s') \\
J_6 &= \int_{-\infty}^{+\infty} ds \left(\frac{dh(s)}{ds}\right)^2 \int_{-\infty}^{s} ds' \int_{-\infty}^{s'} ds'' h(s'')
\end{aligned} \tag{19}
$$

The integrands in Eq. 20 also go rapidly to zero at both sides. The intgeral J_1 is divergent in the sharp-cutoff approximation. There is only one term that contains it,

$$
\begin{aligned}
U_{4333} &= -\frac{J_1\sec^3\beta}{\epsilon\, 2} \\
&\quad - \frac{\sin\beta}{12\cos^5\beta}\left[\sin^2\beta\left(1+6\cos^2\beta\right) + 6J_2 - 42J_4\right] + \cdots
\end{aligned}
$$

Space considerations do not permit to include the complete set of the 34 third-order matrix elements. As a sample, we give the chromatic terms,

$$
\begin{aligned}
U_{1116} &= \frac{\tan^2\beta}{2} + \cdots \\
U_{1336} &= -\frac{\sec^2\beta}{2} + \epsilon I_2 \frac{\sin\beta(5+\sin^2\beta)}{\cos^4\beta} + \cdots \\
U_{2126} &= -\tan^2\beta + \cdots \\
U_{2166} &= \tan\beta + \cdots \\
U_{2336} &= -\frac{\sin\beta(1+\sin^2\beta)}{\cos^3\beta} + \cdots \\
U_{2346} &= \tan^2\beta - \epsilon I_2 \frac{2\sin\beta(1+\sin^2\beta)}{\cos^4\beta} - \cdots \\
U_{3136} &= -\tan^2\beta + \epsilon I_2 \frac{2\sin\beta(1+\sin^2\beta)}{\cos^4\beta} + \cdots \\
U_{4146} &= \tan^2\beta - \epsilon I_2 \frac{2\sin\beta(1+\sin^2\beta)}{\cos^4\beta} + \cdots \\
U_{4236} &= \sec^2\beta - \epsilon I_2 \frac{2\sin\beta(5+\sin^2\beta)}{\cos^4\beta} + \cdots \\
U_{4366} &= -\tan\beta + \epsilon I_2 \frac{3(1+\sin^2\beta)}{\cos^3\beta} + \cdots
\end{aligned}
$$

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