PRECISE DETERMINATION OF THE AMPLITUDE FUNCTIONS IN COLLIDER INSERTIONS WITH AN APPLICATION TO LEP

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Abstract: An accurate knowledge of the beam sizes at collision points is of vital interest for several important aspects of collider's operation. No device is usually available to provide direct information and one relies on the scaling of beam size measurements performed elsewhere, using an acquired knowledge of amplitude functions. This paper reviews two methods often used to establish ß* at the collision point, appraises their performances (with numerical applications to the LEP) and proposes an additional procedure to get ten times more accurate results.

1. Introduction

Most colliders do not have detectors at collision points to directly measure the beam sizes, despite of their importance for the optimization of machine luminosity, for the interpretation of beam-beam effects, for the normalization of cross-sections, etc. The best estimates for $\sigma^*_{x,z}$ are obtained from beam sizes $\sigma_{x,z}$ measured elsewhere in the machine (with a wire scanner or with a synchrotron radiation telescope) and scaled to the collision point by use of the betratron amplitude functions:

$$\sigma^*_{x,z} = \sqrt{\beta^*_{x,z}/\beta_{x,z}}\; \sigma_{x,z} \tag{1}$$

Among the factors in Equ. 1, $\beta^*_{x,z}$ is the most difficult to establish with accuracy. In this paper three measuring procedures are examined and qualified for their achievements. The first one, described in para. 2, relies on measuring the amplitude of coherent oscillations at a close by monitor, and gives good results in the defocussing plane only. In a second procedure the Q-shift is measured which results from a known variation of strength in one of the insertion quads (see para. 3). The accuracy of this well known method is severly limited for practical reasons. Therefore a refined procedure, involving antisymmetric perturbations of both insertion quads, is analysed in para. 4 and leads to far more accurate values for β^*_z (focussing plane).

2. Determination of ß* from a measurement of coherent oscillation amplitude

Beam position monitors can be used to settle the local amplitude functions in a procedure where coherent oscillations of a known amplitude are measured for different phases, thus eliminating the closed orbit. The monitors closer to the collision point must be used, in order to avoid uncertainties in modelling the amplitude function to the collision point. In the absence of beam-beam force the amplitude function ß(s) in the drift space inbetween the insertion quads can be expressed in terms of its minimum value β_m and of an asymmetry parameter ϵ:

$$\beta(\pm\ell) = \beta_m + (\pm\ell-\epsilon)^2/\beta_m \tag{2}$$

The two Equs. 2 can be solved for ϵ and β_m, and since $\beta^* = \beta_m + \epsilon^2/\beta_m$ (see Fig. 1) one gets, after some algebraic manipulations

$$\beta^* = \frac{1}{4}\{\xi - (\xi^2 - 16\ell^2 - \eta^2)^{1/2}\} \tag{3}$$

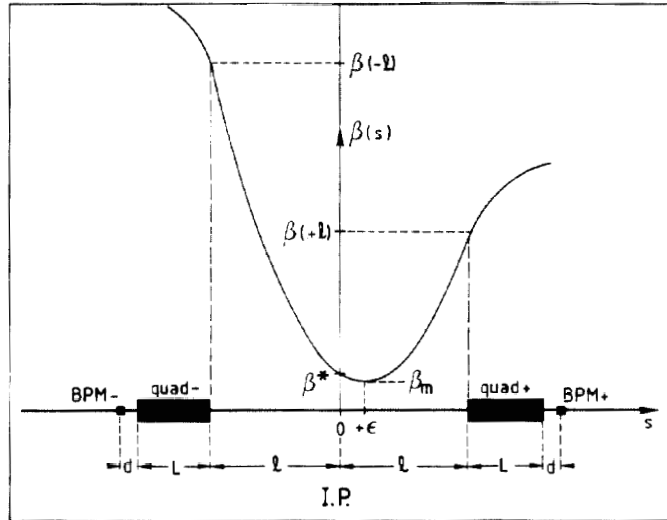with $\xi = \beta(+\ell) + \beta(-\ell)$, $\eta = \beta(+\ell) - \beta(-\ell)$.

Fig. 1 : Layout around a collision point

Let us see now the error propagation from the measured values of ß($\pm\ell$) to the wanted ß*. If $\langle\delta\beta\rangle$ is the r.m.s. measurement error, one has

$$\langle\delta\xi\rangle = \langle\delta\eta\rangle = \sqrt{2}\,\langle\delta\beta\rangle$$

and since ξ and η are statistically independent it follows that

$$\langle\delta\beta^*\rangle = \sqrt{2}\,\{(\partial\beta^*/\partial\xi)^2 + (\partial\beta^*/\partial\eta)^2\}^{1/2}\,\langle\delta\beta\rangle. \tag{4}$$

The partial derivatives in Equ. 4 can be derived from Equ. 3 and finally the error sensitivity is demonstrated by the coefficient κ relating the relative errors:

$$\frac{\langle\partial\beta^*\rangle}{\beta^*} = \kappa\,\frac{\langle\partial\beta\rangle}{\beta(\ell)}$$

Table 1 : Numerical application to LEP

Plane	focussing (z)	defocussing (x)
ß*	0.07 m	1.75 m
ß(ℓ)	195.5 m	9.6 m
ξ	391 m	19.2 m
η	2.8 m	0.14 m
$\delta\beta^*/\delta\xi$	1.86×10^{-4}	0.142
$\delta\beta^*/\delta\eta$	1.79×10^{-3}	0.003
κ	7.15	1.10

Table 1 illustrates the case of LEP where $\ell = 3.7$ m. With the assumption that ß(ℓ) can be measured to 1 %, the smallest controlable value for the beta difference is $\eta \approx \langle\delta\eta\rangle = \sqrt{2}\,\beta(\ell)/100$.

The result is very surprising: in the focussing plane the relative error is increased by nearly an order of magnitude, just because the symmetry of the amplitude function around the crossing point cannot be guarantied!

In practice the numerical application given above is of academic interest because no position monitor exists on the inside the LEP insertion quads. But a similar derivation can be done for BPMs located beyond the quads (see Appendix A) and the resulting accuracy on β^*_z is even worse [1].

3. Determination of β_z^* from the Q-shift induced by quadrupole gradient perturbation

When a small change of strength Δk is applied to one of the insertion quads the resulting Q-shift is given to first order by [2]

$$\Delta Q = \frac{\Delta k}{4\pi} \int_{quad} \beta(s)\,ds$$

or

$$\bar{\beta} = \frac{4\pi\,\Delta Q}{L\,\Delta k} , \qquad (5)$$

with L the magnetic length of the quad. In most cases this measurement is not accurate in the defocussing plane due to the small induced Q-shift and will be disregarded in the following discussion. Let us apply it in the focussing plane to either quad and obtain two independent values called $\bar{\beta}_+$ and $\bar{\beta}_-$. We are now back to the problem of para. 2 of computing β^*. The exact solution is derived in Appendix B and reads in its simplified form (valid for all practical cases where $\bar{\beta} \gg 2\ell$):

$$\beta_z^* \approx \frac{N}{\xi} [1 + \eta^2/P^2] \qquad (6)$$

where $\xi = \bar{\beta}_+ + \bar{\beta}_-$, $\eta = \bar{\beta}_+ - \bar{\beta}_-$, and $N \approx 2\ell^2$ and $P \approx 4\ell$ are two constants. Equ. 6 reveals very clearly the respective influences of ξ and η on the amplitude function at the collision point.

The accuracy of the result can be analysed as follows. For a value of $\bar{\beta}$ one has to measure twice Q and k to get ΔQ and Δk used in Equ. 5. Since all measurement errors involved are independent, they can be added in quadrature to give

$$<\delta\bar{\beta}> = \bar{\beta}\,[2\,(\frac{<\delta k>}{\Delta k})^2 + 2\,(\frac{<\delta Q>}{\Delta Q})^2 + (\frac{<\delta L>}{L})^2]^{1/2} \qquad (7)$$

where $<\delta k>$, $<\delta Q>$ and $<\delta L>$ are the r.m.s. errors. The error propagation is now the same as in para. 2 where only β must be replaced by $\bar{\beta}$ and the partial derivatives of Equ. 6 have to be introduced into Equ. 4.

Numerical application to LEP : The best feasible measuring accuracies are taken as:
- Quadrupole strength : $<\delta k>/k = 2 \times 10^{-5}$
- Magnetic length : $<\delta L>/L = 10^{-3}$
- Q measurement : $<\delta Q> = 2 \times 10^{-4}$.

Simulations with MAD [3] show that for $\Delta k = 10^{-3}$ the Q-shift ($\Delta Q_z = 0.0413$) is at the limit of linearity for Equ. 5 and should also just be acceptable in the working diamond. $\Delta k = 10^{-3}$ is used for computing the best performances shown in Table 2. Other relevant parameters are [4]: $\ell = 3.7$ m, $L = 2$ m, $k = 0.1646$ m^{-2}, for the insertion with nominal $\beta_x^* = 1.75$ m and $\beta_z^* = 7$ cm.

In Table 2 the different lines correspond to values of the unbalance parameter $\eta = \bar{\beta}_+ - \bar{\beta}_-$ chosen arbitrarily but always larger than the measuring accuracy $<\delta\eta> = 3.1$ m obtained from Equ. 7.

Table 2 : Performances for β_z^* measurements

ξ	η	ε	β_z^*	$<\delta\beta_z^*>/\beta_z^*$
m	m	cm	cm	%
518	3	1.4	7.3	7.7
518	5	2.3	7.8	12.0
518	7	3.2	8.5	15.4
518	10	4.6	10.1	18.6
518	20	9.3	19.2	19.5

Because of the very low beta, the sensitivity to the exact waist position is extreme and values of $\eta \gtrsim 4$ m should be avoided by all means. The errors, shown in column 5, mainly result from the second term in Equ. 4 and would reduce if only η could be measured with a greater accuracy. This is done in the next paragraph.

4. The antisymmetric gradient perturbation method

In order to apply higher gradient perturbations without boosting ΔQ_z beyond 0.04, one can apply simultaneously $+\Delta k$ on one quad and $-\Delta k$ on the other quad of the insertion. The observed tune will then be :

$$Q_1 = Q_0 + \frac{L\eta}{4\pi} [\Delta k + \alpha\,\Delta k^2 + O(\Delta k^3)] \qquad (8)$$

where Q_0 corresponds to the unperturbed case, $\eta = \bar{\beta}_+ - \bar{\beta}_-$, $\alpha\,\Delta k^2$ and $O(\Delta k^3)$ are higher order terms in Δk. If the signs of the perturbations are reversed, the measured tune becomes :

$$Q_2 = Q_0 - \frac{L\eta}{4\pi} [\Delta k - \alpha\,\Delta k^2 + O(\Delta k^3)] \qquad (9)$$

and the difference $\Delta Q = Q_1 - Q_2$ reads :

$$\Delta Q = \frac{L\eta}{2\pi} [\Delta k + O(\Delta k^3)] \qquad (10)$$

In Equ. 10, Q_0 has disappeared and the second order terms vanish so that the linear expression holds up to third order in Δk and provides for an accurate determination of η. Simulations made with MAD [3] have shown that the linearity is preserved up to $\Delta k \le 5\ 10^{-3}$. Therefore using antisymmetric perturbations leads through Equ. 10 to values of η 20 times more precise than achieved in para. 3 since $<\delta\eta>$ now reads:

$$<\delta\eta> = \frac{2\sqrt{2}\pi\,<\delta Q>}{L\,\Delta k}$$

This result can be used to get a much higher accuracy in the determination of β_z^* given by Equ. B6, when the value of ξ is obtained by the traditional method of para. 3 and the value of η is given by Equ. 10.

The accuracies on β_z^* got by both methods are shown in Fig. 2 versus η, for three values of the Q-measuring resolution $<\delta Q>$.
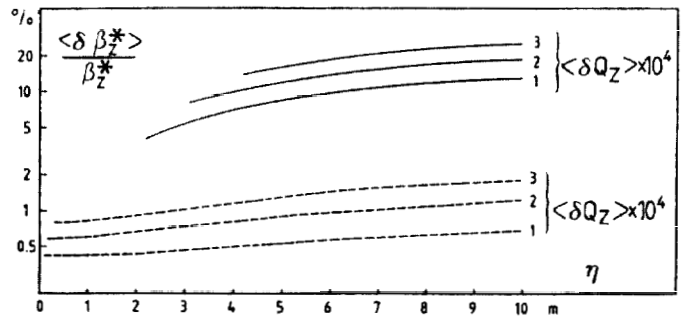

Fig. 2: Accuracy on β_z^* from : ——— method of para. 3 ------- method of para. 4

5. Conclusions

The three procedures discussed above are fully complementary to provide β^* with the best accuracy. They all rely, in the analytic derivation, on the assumption that there is no beam-beam effect and should thus preferably be performed with a single beam.

References

[1] C. Bovet, C. Fischer, "Precise Determination of Beta Values at LEP Collision Points", LEP Note 612 (1988).

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[3] F.C. Iselin, "The MAD Program", CERN-LEP/TH-88-38 (July 1988).

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[5] C. Bovet et al., "A Selection of Formulae and Data useful for the Design of A.C. Synchrotrons", CERN/MPS-SI/Int. DL/70/4.

[6] S. Keifets, "Simple Formulae for β Function", PEP Note 337, (September 1980).

APPENDIX A : Propagation of the beta functions from the waist to the nearest BPM's

Let the waist near a collision point be characterised by β_m and ϵ (see Fig. 1). The transfer matrix from the waist to the BPM reads :

$$M(s_2/s_1) = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 & \ell \mp \epsilon \\ 0 & 1 \end{pmatrix} \quad (A1)$$

with ϵ, ℓ and d defined in Fig. 1 and a_{ij} the quad matrix elements.

Twiss parameters transform through M like [5]:

$$\beta_2 = m_{11}^2 \beta_1 - 2 m_{11}m_{12} \alpha_1 + m_{12}^2 \gamma_1 \quad (A2)$$

Since we have $\beta_1 = \beta_m$, $\alpha_1 = 0$ and $\gamma_1 = 1/\beta_m$, we can write the betas at the two BPM's as:

$$\beta_\pm = A^2 \beta_m + [A(\ell \mp \epsilon) + B]^2/\beta_m \quad (A3)$$

with $A = a_{11} + a_{21}d$ and $B = a_{12} + a_{22}d$. Using again $\xi = \beta_+ + \beta_-$ and $\eta = \beta_+ - \beta_-$ the two Equs. A3 become

$$\eta = -4A\epsilon (\ell A + B)/\beta_m \quad (A4)$$

which reduces to:

$$\epsilon = \frac{\eta \beta_m}{E} \quad (A5)$$

with $E = -4A(\ell A + B)$ and:

$$\xi = 2 A^2 \beta_m + 2[\ell A + B]^2/\beta_m + 2 A^2 \epsilon^2/\beta_m \quad (A6)$$

Equ. A6 can be solved for β_m after substitution of ϵ by use of Equ. A4 and reads:

$$\beta_m = \frac{\xi - [\xi^2 - E^2 - \eta^2]^{1/2}}{4 A^2 (1 + \eta^2/E^2)} \quad (A7)$$

The final result for $\beta*$ is:

$$\boxed{\beta^* = \frac{\xi - [\xi^2 - E^2 - \eta^2]^{1/2}}{4 A^2}} \quad (A8)$$

APPENDIX B : Expressions for β_m, ϵ and β_z^* in terms of $\bar{\beta}_+$ and $\bar{\beta}_-$

The variation of $\beta(s)$ inside a focussing quadrupole can be expressed as [6]:

$$\beta(s) = \frac{\beta_o}{2} (1+\cos \omega s) + \frac{\beta_o'^2+4}{2\omega^2\beta_o} (1-\cos \omega s) + \frac{\beta_o'}{\omega} \sin \omega s \quad (B1)$$

where the origin of s is taken at the quadrupole entrance, β_o and β_o' are initial conditions, and $\omega^2 = 4k$.

The average value $\bar{\beta}$ obtained by integration of Equ. B1 over the quadrupole length, L, is:

$$\bar{\beta} = \frac{\beta_o}{2} (1+\frac{\sin u}{u}) + \frac{\beta_o'^2+4}{2\beta_o\omega^2} (1-\frac{\sin u}{u}) - \frac{\beta_o'}{\omega} (\frac{\cos u}{u} - \frac{1}{u}) \quad (B2)$$

with $u = \omega L$.

Let us now express the initial conditions in terms of the beta function existing around the collision point and given by Equ. 2:

$$\beta_{o\pm} = \beta_m + \frac{(\ell \mp \epsilon)^2}{\beta_m}$$
$$\beta_{o\pm}' = \frac{2(\ell \mp \epsilon)}{\beta_m} \quad (B3)$$

Note that on either side one gets

$$\frac{\beta_o'^2 + 4}{\beta_o} = \frac{4}{\beta_m}$$

The difference and the sum of the average betas measured on both quadrupoles can be expressed, using Equ. B2 and Equ. B3 :

$$\eta = \bar{\beta}_+ - \bar{\beta}_- = -\frac{2\ell\epsilon}{\beta_m} (1 + \frac{\sin u}{u}) + \frac{4\epsilon}{\omega\beta_m} (\frac{\cos u - 1}{u})$$

which reduces to

$$\epsilon = \frac{\eta \beta_m}{P} \quad (B4)$$

with $P = -2\ell(1 + \frac{\sin u}{u}) + \frac{4}{\omega}(\frac{\cos u - 1}{u})$; and also :

$$\xi = \bar{\beta}_+ + \bar{\beta}_- = \frac{\ell^2 + \beta_m^2 + \epsilon^2}{\beta_m} (1 + \frac{\sin u}{u}) +$$
$$+ \frac{4}{\omega^2\beta_m} (1 - \frac{\sin u}{u}) - \frac{4\ell}{\omega\beta_m} (\frac{\cos u - 1}{u}),$$

which can be expressed, after substitution of Equ. B4 for ϵ, as:

$$M(1 + \eta^2/P^2) \beta_m^2 - \xi\beta_m + N = 0 ,$$

with $M = 1 + \frac{\sin u}{u}$

$$N = \ell^2(1+\frac{\sin u}{u}) + \frac{4}{\omega^2} (1-\frac{\sin u}{u}) - \frac{4\ell}{\omega} (\frac{\cos u - 1}{u})$$

and the solution for β_m is :

$$\beta_m = \frac{\xi - [\xi^2 - 4 M N (1 + \eta^2/P^2)]^{1/2}}{2 M(1 + \eta^2/P^2)} \quad (B5)$$

Since $\beta^* = \beta_m + \epsilon^2/\beta_m$, using Equ. B4 one has

$$\beta_z^* = \beta_m (1 + \eta^2/P^2)$$

so that finally

$$\boxed{\beta_z^* = \frac{\xi - [\xi^2 - 4 M N (1 + \eta^2/P^2)]^{1/2}}{2 M}} \quad (B6)$$

Since $\xi^2 \gg 4 M N$, Equ. B5 and B6 can be simplified by expanding the square root to first order and one gets

$$\beta_m \approx \frac{N}{\xi}$$

and

$$\boxed{\beta_z^* \approx \frac{N}{\xi} (1 + \eta^2/P^2)} \quad (B7)$$